MAT2041 **TA: Jiaying Liao, Bowen Kuang**

Assignment 1: Vectors and Vector Operations

Assigned: Sep/13/23 Wednesday Due: Sep/20/23 Wednesday at 11:59pm

Rules (please read!!)

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. **Be sure to submit your homework as a single file.**
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you. However, only insightful discussions are allowed. Directly sharing the solutions is prohibited.
- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments!

Total: 110 points, 7 problems.

1. Let $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ be the dot-product of $\boldsymbol{v}, \boldsymbol{w}$. Verify that the following properties hold using the definition of dot-products (20pts):

a) Linearity: $\langle a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w} \rangle = a\langle \boldsymbol{u}, \boldsymbol{w} \rangle + b\langle \boldsymbol{v}, \boldsymbol{w} \rangle$. (5pts)

b) Symmetric Property: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$. (5pts)

c) Positive Definite Property: For any $\boldsymbol{u} \in V$, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$; and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ if and only if $\boldsymbol{u} = 0$. (5+5pts)

2. Verify that the norm defined in the lecture is indeed a "norm", i.e., it satisfies the following (20pts)

N1: $\|\boldsymbol{v}\| \geq 0$ (5pts) and $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = 0$ (5pts).

N2: $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$ (5pts).

N3: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (5pts).

3. *Reverse triangle inequality*. Suppose $\mathbf{a}$ and $\mathbf{b}$ are vectors of the same size. The triangle inequality

states that $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\|+\|\mathbf{b}\|$. Show that we also have $\|\mathbf{a}+\mathbf{b}\| \geq \|\mathbf{a}\|-\|\mathbf{b}\|$. Hints. Draw a picture to get the idea. To show the inequality, apply the triangle inequality to $(\mathbf{a}+\mathbf{b})$ and $(-\mathbf{b})$. (10pts)

4. *Transforming between two encodings for Boolean vectors.* A Boolean n-vector is one for which all entries are either 0 or 1. Such vectors are used to encode whether each of n conditions holds, with $a_i = 1$ meaning that condition i holds. Another common encoding of the same information uses the two values -1 and $+1$ for the entries. For example the Boolean vector $(0, 1, 1, 0)$ would be written using this alternative encoding as $(-1, +1, +1, -1)$. Suppose that x is a Boolean vector with entries that are 0 or 1, and y is a vector encoding the same information using the values -1 and $+1$. Express y in terms of x using vector notation. Also, express x in terms of y using vector notation. (15pts)

5. *Symptoms vector.* A vector s records whether each of 30 different symptoms is present in a medical patient, with $s_i = 1$ meaning that a patient has the corresponding symptom and $s_i = 0$ meaning she does not. Express the following using vector notation. (10pts)

a) The total number of symptoms the patient has. (5pts)

b) The patient exhibits 6 out of the first 15 symptoms. (5pts)

Remark: For example, the patient exhibits the 1st, 2nd, 5th, 8th, 10th, 14th, and 26th symptoms but not other symptoms. As another example, the patient exhibits the 2nd, 6th, 7th, 8th, 10th, 11th, 18th and 27th symptoms but not other symptoms.

Hint: (a) Define a vector, and use an inner product. (b) Use an equation.

6. *Interpretation of angles.* Denote $\angle(\mathbf{u}, \mathbf{v})$ as the angle between any two vectors $\mathbf{u}$ and $\mathbf{v}$. Suppose there are three types of fruit in the super market: orange, apple and banana. Every week, Alice buys 2 kilos of oranges, 1 kilos of apples and 5 kilos of bananas; Bob buys 5 kilos of oranges, 9 kilos of apples and 1 kilos of bananas; Charlie buys 3 kilos of oranges, 1 kilos of apples and 2 kilos of bananas. The three vectors corresponding to their choices are $\mathbf{a} = (2, 1, 5)$, $\mathbf{b} = (5, 9, 1)$, $\mathbf{c} = (3, 1, 2)$. (15pts)

i) Compute the two angles $\theta_1 = \angle(\mathbf{a}, \mathbf{b})$ and $\theta_2 = \angle(\mathbf{a}, \mathbf{c})$. Make sure to convert the angles to degree measurements (for example, write 90° instead of $\frac{\pi}{2}$). (5pts)

Hint: You can use the website Wolfram Alpha or Symbolab to compute the arccos function; see `https://www.wolframalpha.com/input?i=arc+cos+0` or https://zs.symbolab.com/solver/step-by-step/%5Carccos%5Cleft(0%5Cright)?or=input.

ii) Which one is larger? Provide an interpretation of this comparison. (5pts for correct interpretation, a correct guess without interpretation will receive 0)

Hint: The keyword you can use is "preference" or "taste". Use your own words to describe the interpretation. The answer can be a few sentences.

iii) In general, when evaluating an object (e.g. a university, a restaurant, a cellphone), everyone has their own "weight vector". Provide an interpretation of the angle between the weight vectors of two people. (5pts)

7. *Matrix multiplication.* Given an $n \times n$ matrix A and an n-dimensional vector x, we want to calculate the total number of multiplications (only the number of mutiplications and ignore other operations such as additions) to compute the matrix-vector product Ax. (20pts)

i) Show that we can compute Ax by n^2 multiplications. (5pts)

ii) We say A is of rank one if $A = uv^T$, where u and v are n-dimensional column vectors and v^T is the transpose of v, show that we can compute Ax by $2n$ multiplications. (5pts)

iii) We say A is of rank at most r (assuming $r < n$) if A can be written as the sum of r rank-one matrices, show that for such A we can compute Ax by $2rn$ multiplications. (10pts)