MAT2041 **TA: Jiaying Liao, Bowen Kuang**

Assignment 2: System of Linear Equations, Matrix Multiplication

Assigned: Sep/28/23 Thursday Due: Oct/09/23 Monday at **10:00pm**

Rules (please read!!)

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. **Be sure to submit your homework as a single file.**
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you.

  However, only insightful discussions are allowed. Directly sharing the solutions is prohibited.
- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments!

Total: 90 points, 7 problems.

Problem 1. ***(Solve linear equations)*** ($5+5$ = 10 points)

Represent the linear systems with augmented matrices, solve the following systems and write down the solution.

(a) $$\begin{cases} 2x_1 + 2x_2 + 3x_3 = -7 \\ 2x_1 + x_2 + 3x_3 = 6 \\ x_1 + 2x_2 + x_3 = -4 \end{cases}$$

(b) $$\begin{cases} -x_1 + 0x_2 - x_3 - 2x_4 = -4 \\ x_1 + x_2 - x_3 - 2x_4 = 0 \\ 0x_1 - x_2 + x_3 + 2x_4 = 1 \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

Problem 2. ***(Interpolation of rational functions)*** (15 points)

A rational function of degree two has the form

$$f(t) = \frac{c_1 + c_2 t + c_3 t^2}{1 + d_1 t + d_2 t^2}$$

where c_1, c_2, c_3, d_1, d_2 are coefficients. (Rational refers to the fact that f is a ratio of polynomials. Another name for f is bi-quadratic.) Consider the interpolation conditions

$$f(t_i) = y_i,\ i = 1, 2, ..., K$$

where t_i and y_i are given numbers. Express the interpolation conditions as a set of linear equations in the vector of coefficients $\theta = (c_1, c_2, c_3, d_1, d_2)$, as $A\theta = b$. Give A and b, and their dimensions.

Problem 3. ***(Matrix products)*** (5 + 5 = 10 points)

(i) Suppose $\mathbf{U}$ and $\mathbf{R}$ are $n \times n$ upper triangular matrices and $\mathbf{T} = \mathbf{UR}$, show that $\mathbf{T}$ is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for all $j = 1, ..., n$.

Remark: In the tutorial session, we have proved this conclusion by block-matrix multiplication. One will receive **0 point** if he/she uses the **same method** provided in the tutorial. Alternatively, to receive full points, the **conventional matrix multiplication** should be used.

(ii) Suppose $\mathbf{A}$ is an upper triangular $n \times n$ matrix and $\mathbf{B}$ is an arbitrary $n \times n$ matrix. Denote $\mathbf{C} = \mathbf{AB}$. Alice conjectures that $c_{jj} = a_{jj}b_{jj}$ for all $j = 1, ..., n$. Is this conjecture true? If yes, prove it; if no, provide a counter-example.

Remark: Counter-example is an example such that the conjecture does not hold. In this problem, "providing a counter-example" means finding two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfying the requirements such that the statement "$c_{jj} = a_{jj}b_{jj},\ \forall j \in \{1, 2, ..., n\}$" does not hold.

Problem 4. ***(products of partitioned matrices)*** (9 + 6 = 15 points)

(a) Suppose $\mathbf{v} \in \mathbb{R}^{m\times 1}$ and $A \in \mathbb{R}^{m\times n}$. The product $\mathbf{v}^\top A$ is a $1 \times n$ vector. Express $\mathbf{v}^\top A$ in three forms: scalar form (the expression that involves the entries of A), the form that involves the rows of A, and the form that involves the columns of A.

(b) Suppose $A = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$. Write the expressions of $AA^\top$ and $A^\top A$ using $\mathbf{a}_1, \dots, \mathbf{a}_n$.

Problem 5. ***(Matrix Multiplication)*** (15 points)

For two square matrices A and B, show that if $AB = A + B$, then $BA = A + B$.

Hint: You can use the following conclusion without proof:

For two square matrices C and D, if $CD = I$, then $DC = I$.

Problem 6. ***(LU).*** (5 + 5 = 10 points)

Find the LU decomposition of the following two matrices.

(a)

$$A_1 = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 13 & 21 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$A_2 = \begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & -80 \end{bmatrix}.$$

Problem 7 (Judgement) (3 × 5 = 15 points)

Judge whether each statement is true or false. If true, provide a brief explanation why it is true. **If false, provide a counter-example** (i.e., a specific example that the statement does not hold)

(a) If a square matrix A satisfies $A^2 = 0$, then A is a zero matrix.

(b) Suppose 2×2 square matrices A, B, C satisfy $AB = AC$, then we must have $B = C$.

(c) For any $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$ where $k \geq 3$, if the first column and the third column of B are the same, then the first column and the third column of AB are the same.

(d) For any $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$ where $n \geq 3$, $m \geq 3$, if the first row and the third row of matrix B are the same, then the first row and the third row of AB are the same.

(e) There exists a matrix $A \in \mathbb{R}^{m\times n}$ such that for any matrix $B \in \mathbb{R}^{n\times k}$, every row of AB is the same as the first row of B.

Remark: For easier grading, you are suggested to write the solution as follows.

" (a) True.

Reason: "

or " (b) False.

Counter-example: "