MAT2041 **TA: Yifei Wu, Yijie Zhou**

HW 3: Matrix Operations, Inverse, and LU

Assigned: Oct/12/23 Thursday Due: Oct/22/23 Sunday at 10:00pm

Rules

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. Be sure to submit your homework as a single readable file.
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you.

  Notably, only insightful discussions are allowed. Directly sharing the solutions is prohibited. The details of the collaboration policy for this course are available in the Resources tab on Piazza.
- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments.

Total: 140 pts, 7 problems

Problem 1 (Solve Linear Equations) *(5 + 5 = 10 pts)*

Solve the following linear systems using the Gaussian elimination. You **must** represent the systems with augmented matrices, transform them to the reduced row-echelon form and write down the solution sets.

Remark: You get 1 mark for the augmented matrix, 2 for the RREF and 2 for the solution set.

(a) $\begin{cases} 2x_1 - x_2 = 1 \\ x_1 + x_2 = 2 \\ 5x_1 - x_2 = 4 \end{cases}$

(b) $\begin{cases} 3x_1 - x_3 - 2x_4 = -4 \\ 3x_1 + 3x_2 - x_3 - 2x_4 = 0 \\ -3x_2 + x_3 + 2x_4 = 1 \end{cases}$

Problem 2 (Short-Answer Questions) *(2 + 3 + 3 + 3 + 3 + 3 + 3 = 20 pts)*

Solve the following problems and justify your answers by showing your work

(a) Find the inverse of $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

(b) Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.

(c) Give example of 2×2 non-zero matrices A, B, C such that $AB = AC$ but $B \neq C$

(d) Write 2×2 invertible matrices A and B such that $A + B$ is not invertible.

(e) **T** or **F**? If A and B are invertible matrices of same size, then AB and BA are both invertible.

(f) **T** or **F**? Any square matrix A can be written as a product $A = LU$ for a certain lower triangular matrix L and a certain upper triangular matrix U.

(g) **T** or **F**? If $A^3 = 2I$, $B = A^2 - 2A + 2I$ then B is invertible, and $B^{-1} = \frac{1}{10}(A^2 + 3A + 4I)$. [Hint: $A^3 = 2I, A$ is invertible]

Remark: For judgement questions, you are suggested to write the solution as follows. “ (a) True. Reason: ” or “ (a) False. Counter-example: ”

Grading scheme: 1 point for correct judgement, and 2 points for reasons or counter-examples.

Problem 3 (Linear Systems) *(5 + 5 + 5 = 15 pts)*

Given the linear system

$$\begin{cases} 2x + 4y - 3z = 7 \\ 8x + 17y - 3z = 39 \\ ax - 3y + 7z = -2 \end{cases} .$$

(a) Suppose $a = -3$. Solve by reducing into upper triangular form and back substitution, and list all multipliers used and circle all the pivots.

(b) Reduce the coefficient matrix into upper triangular form. Determine the value of 'a' for which the elimination fails to give pivots.

(c) Under the assumption of (a), compute LU decomposition for coefficient matrix.

Problem 4 (LU decomposition of Vandermonde Matrix) *(6 + 16 + 8 = 30 pts)*

In HW2, we consider interpolation problem for rational functions. Here we continue with a simpler problem of polynomial interpolation.

Given n distinct real numbers $x_1, x_2, ..., x_n$, and a polynomial $p(t) = a_0 + a_1 t + ... + a_{n-1} t^{n-1}$. Also given a set of real values $b_1, \dots, b_n$. Find $p(x)$, i.e., find $a_0, a_1, \dots, a_{n-1}$ such that $p(x_i) = b_i, i = 1, ..., n$.

(a) Show that the above problem can be reduced to solving the following linear system:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

(b) Consider the coefficient matrix of the above linear system (called Vandermonde matrix)

$$V_n(x_1, \dots, x_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

To simplify, let us denote $V_n = V_n(x_1, \dots, x_n)$.

Use the LU decomposition to prove: if $x_i, i = 1, 2, ..., n$ are distinct (meaning that $x_i \neq x_j$ for any $i \neq j$), then the coefficient matrix $V_n(x_1, \dots, x_n)$ is invertible.

[1] One way to prove the result is to use determinant, a concept that we will learn later. The determinant of the Vandermonde matrix can be expressed as

$$\det(V_n(x)) = \prod_{1 \le i \le j \le n} (x_i - x_j).$$

Therefore, if the numbers $x_1, x_2, ..., x_n$ are distinct, V is a nonsingular matrix. However, you should NOT use this method for this problem.

[1]. Hint: You can use elimination to prove the following relation

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -x_1 & 1 & \cdots & 0 \\ 0 & -x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -x_{n-1} & 1 \end{bmatrix} V_n(x_1, \ldots, x_n) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V_{n-1}(x_2, \ldots, x_n) \end{bmatrix} \left[\begin{array}{c|cccc} 1 & & \mathbf{1} & & \\ \hline & x_2 - x_1 & & & \mathbf{0} \\ \mathbf{0} & & x_3 - x_2 & & \\ & & & \ddots & \\ & \mathbf{0} & & & x_n - x_{n-1} \end{array}\right].$$

(c) Continue (b)'s solution, given an explicit form of upper triangular matrix U^{-1}.

[Hint 1: Write U as a product of n upper triangular matrices.]

[Hin 2: Can use the conclusions of Problem 6.]

Problem 5 ($PA = LU$) *(5 + 5 + 5 = 15 pts)*

(a) Find the inverse of a 3 by 3 lower triangular matrix L, with nonzero entries a, b, c, d, e, f. You could use Gauss-Jordan elimination[2] for the inverse. Find the inverse of

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}.$$

(b) Compute an $PA = LU$ decomposition of

$$A = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

(c) Under the assumption of (b), use the above $PA = LU$ results to find the solution of the following two systems $Ax = \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$ and $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Problem 6 (Sherman-Morrison Formula) *(6 + 4 + 4 + 6 + 10 = 30 pts)*

Suppose $A \in \mathbb{R}^{n \times n}$ is invertible and $u, v \in \mathbb{R}^{n \times 1}$. Check and verify the following formulas.

(a) Let $B = A + uv^\top$ and assume $1 + v^\top A^{-1} u \neq 0$, verify that

$$B^{-1} = A^{-1} - \frac{A^{-1} u v^\top A^{-1}}{1 + v^\top A^{-1} u}.$$

(b) Show that B is invertible if and only if $1 + v^\top A^{-1} u \neq 0$.

[2]Gaussian-Jordan elimination is another version of Gaussian elimination, except that it reduces the matrix to a simplified row-ladder form. Please check `https://brilliant.org/wiki/gaussian-elimination/#computing-inverses`

(c) Let $C = A + e_k v^T$ and assume C to be invertible, use the Sherman-Morrison formula to express C^{-1} in terms of A^{-1}, v, and e_k.

(d) Let $U \in R^{n\times n}$ be an upper triangular matrix with non-zero diagonals, i.e. $U_{ii} \neq 0$. Define u_i as the i-th cloumn of U, and e_i is the i-th cloumn of the identity matrix I. Verify that

$$U^{-1} = \left(I - \frac{\tilde{u}_1 e_1^T}{U_{11}}\right) \cdots \left(I - \frac{\tilde{u}_{n-1} e_{n-1}^T}{U_{n-1n-1}}\right)\left(I - \frac{\tilde{u}_n e_n^T}{U_{nn}}\right),$$

where $\tilde{u}_i = u_i - e_i$.

[Hint: Reformulate U as a product of matrices, and for each of the factor in the product use the Sherman-Morrison Formula.]

(e) Now consider a perturbation $UV^\top$, where $U \in \mathbb{R}^{n\times k}$ and $V \in \mathbb{R}^{n\times k}$, $k < n$. If $I + V^\top AU$ is nonsingular, verify that $A + UV^\top$ is nonsingular and

$$(A + UV^\top)^{-1} = A^{-1} - A^{-1}U(I + V^\top A^{-1}U)^{-1}V^\top A^{-1}.$$

[Hint: Matrix partitioning can be a good idea.]

Problem 7 (Inverse) (*6 + 6 + 8 = 20 pts*)

Suppose $A, M \in \mathbb{R}^{n\times n}$ are two square matrices.

(a) Suppose $B, C \in \mathbb{R}^{n\times n}$ are invertible matrices satisfying $BA = CM$. Prove: A is invertible iff M is invertible.

(b) Suppose $D, E \in \mathbb{R}^{n\times n}$ are invertible matrices satisfying $DA = ME$. Prove: A is invertible iff M is invertible.

(c) Consider V_n in Problem 4. Prove: If $x_i = x_j$ for some $i \neq j$, then V_n is not invertible.