MAT2041 **TA: Yifei Wu, Yijie Zhou**

HW 4: Linear space, Basis, Dimension, Solving Ax=b

Assigned: Oct/27/23 Friday Due: Nov/2/23 Thursday at 10:00pm

Rules

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. Be sure to submit your homework as a single readable file.
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you.

  Notably, only insightful discussions are allowed. Directly sharing the solutions is prohibited. The details of the collaboration policy for this course are available in the Resources tab on Piazza.
- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments.

Total: 70+10 points, 8 problems

1. (Short-Answer Questions) (4 × 2 = 8 points)

   Judge whether each statement is true or false.

   (a) The vectors $\mathbf{v}_1 = (1, 1, 1), \mathbf{v}_2 = (3, 8, 0), \mathbf{v}_3 = (1, 2, -1)$ span $\mathbb{R}^3$.

   (b) The vector $\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ is in the column space of $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ with coefficients $\frac{2}{7}, \frac{1}{7}, \frac{16}{7}$.

   (c) If the columns of a matrix are linearly dependent, then the rows of this matrix must also be linearly dependent.

   (d) Suppose $V_p = \mathbb{R}^2 = \{(x, y, z)|2x + 3y - 2z = 0, x, y, z \in \mathbb{R}\}$ represents a plane set, then here exists a basis for the intersection of V_p with yz-plane.

   Remark: For judgement questions, you are suggested to write the solution as follows. " (a) True. Reason: " or " (a) False. Counter-example: "

   Grading scheme: 1 point for correct judgement, and 2 points for reasons or counter-examples.

2. Check whether the following vectors are linearly independent or dependent. If they are linearly dependent, find one vector which can be expressed by remaining vectors and find its expression. (2× 5 = 10 points)

   (a)
$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \boldsymbol{\alpha}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \boldsymbol{\alpha}_3 = \begin{pmatrix} 2 \\ 5 \\ 0 \\ 1 \end{pmatrix}$$

   (b)
$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \boldsymbol{\alpha}_2 = \begin{pmatrix} 5 \\ 4 \\ -3 \\ 1 \end{pmatrix}, \boldsymbol{\alpha}_3 = \begin{pmatrix} 6 \\ 6 \\ -2 \\ 2 \end{pmatrix}, \boldsymbol{\alpha}_4 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}$$

3. Calculate the rank of following matrix. (10 points)
$$A = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 1 & 1 & -2 & 4 \\ -2 & 1 & 0 & 3 \\ 0 & 5 & -4 & 2 \end{pmatrix}$$

4. Let A be an $m \times n$ matrix of rank r. For each pair of values of m, n and r below, how many solutions could one have for the linear system $Ax = b$? Explain your answers. (1 × 4 = 4 points)

   (a) $m = n = r = 6$;

   (b) $m = 9, n = r = 7$;

   (c) $m = r = 7, n = 8$;

(d) $m = n = 8, r = 7$.

5. (Rank) (6 + 6 + 6 = 18 points)

You can use the following result in this problem: Elementary row operation does not change the rank of a matrix.

Let $A \in \mathbb{R}^{n\times n}$, prove:

(a) For arbitrary matrix D, we have rank($\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ $= \text{rank}(A) + \text{rank}(D)$

(b) Let the matrix A be invertible. For arbitrary matrices B, C, D satisfied the the partitioned form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

we can get the equation: rank($\begin{bmatrix} A & B \\ C & D \end{bmatrix}$) = rank(A) + rank($D - CA^{-1}B$).

[Hint: Use block row operation.]

(c) Suppose two matrices $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times p}$. Prove: $\text{rank}(AB) \le \text{rank}(A)$.

[Hint: Consider the row spaces or columns spaces.]

6. (Subspace and Basis) (3 + 2 + 3 + 2 = 10 points)

Let $A \in \mathbb{R}^{n\times n}$.

(a) Suppose $V = \{B \in \mathbb{R}^{n\times n} | AB = BA\}$, Prove that V is a subspace of $\mathbb{R}^{n\times n}$.

(b) If $A = I$, then what is the relationship between V and $\mathbb{R}^{n\times n}$?

(c) If $A = \text{diag}(1, 2, ..., n)$, find a basis for V in (a).

(d) If $n = 3, A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, find a basis for V in (a).

7. (Linear independence) (3 + 3 + 4 = 10 points)

Let $v_1, \cdots, v_n \in \mathbb{R}^m$, prove the following:

(a) if $v_1, \cdots, v_n$ are dependent, then for any $A \in \mathbb{R}^{p\times m}$, $Av_1, \cdots, Av_n$ are dependent.

(b) if $v_1, ..., v_n$ are independent, and $A \in R^{m\times m}$ is nonsingular, then $Av_1, \cdots, Av_n$ are independent.

(c) if $Av_1, \cdots, Av_n$ are independent, then $v_1, \cdots, v_n$ are independent.

The following question is optional. Answering it correctly can gain you some extra grades.

8. (Vandermonde matrix and different interpolation schemes) [BONUS question] Optioinal (5 + 5 = 10 points)

   To interpolate a set of n distinct points at $(r_1, ..., r_n)$, we can do either of the followings:

   - Using monomials $1, t, ..., t^{n-1}$;
   - Using Newton's polynomials $1, \eta_1(t), ..., \eta_{n-1}(t)$, where $\eta_i(t) = \prod_{j=1}^{i}(t - r_j)$;
   - Using Lagrange polynomials $l_1(t), l_2(t), ..., l_n(t)$, where $l_i(t) = \prod_{j=1, j\neq i}^{n} \frac{t-r_j}{r_i-r_j}$.

   Moreover, the last two schemes can be obtained from the first by basis change

$$(1, t, ..., t^{n-1})B = (1, \eta_1(t), ..., \eta_{n-1}(t)), \tag{1}$$

$$(1, t, ..., t^{n-1})C = (l_1(t), l_2(t), ..., l_n(t)). \tag{2}$$

   Find out the expression of B, C as products of matrices.

   Remark: Basis change has not been covered in class. If you don't know what it means, you can skip the problem. If you know what it means and you can solve it, then you can give it a try.

   [Hint: Consider the LU decomposition of the Vandermonde matrix and the properties of the Newton's and Langrange polynomials(when will them be zero). Q4 of the last assignment will help.]