MAT2041 **TA: Hang ZENG, Meng ZHANG**

Assignment 5: Orthogonality, Determinant, and Linear Transformation

Assigned: Nov/14/23 Tuesday Due: Nov 21/23 Tuesday at 10:00pm

Rules (please read!!)

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. **Be sure to submit your homework as a single file.**
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you.

  However, only insightful discussions are allowed. Directly sharing the solutions is prohibited. The details of the collaboration policy for this course are available in the Resources tab on Piazza.
- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments!

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- **Total:** 100 points, 6+1 problems.

1. **Least Squares Problem** (20 points)

   The number of goals, number of assists, and the salary of three soccer players are listed below.

Name	Goals	Assist	Salary (million dollars)
Messi	7	10	30
Haaland	18	3	40
Bruyne	3	10	20

Table 1: Caption

   Suppose another player Tom has 10 goals and 8 assists. Form a least squares problem and provide an estimate of a reasonable salary for Tom. You can use a calculator if needed.

2. **Orthogonality** (10 points)

   Suppose S is a subspace of $\mathbb{R}^n$. Suppose $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{p} \in S$ satisfy $(\mathbf{b} - \mathbf{p}) \perp S$. Prove that for any $\mathbf{z} \in S$, we have $\|\mathbf{b} - \mathbf{z}\|^2 - \|\mathbf{b} - \mathbf{p}\|^2 = \|\mathbf{z} - \mathbf{p}\|^2$.

3. **Orthonormal Basis** $(2 \times 10 = 20$ points$)$

(a) Convert the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\operatorname{span}(B)$ into an orthonormal basis of B, where $\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (1,1,0), \mathbf{u}_3 = (1,0,0)$. Use the Gram-Schmidt Process.

(b) Suppose U is the orthogonal complement of *span* $\left\{ \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \right\}$ in $\mathbb{R}^3$. Find an *orthonormal basis* of U.

4. **Determinant** (10 points)

Suppose $A = \begin{pmatrix} d_1 & a_2 & a_3 & \cdots & a_n \\ b_2 & d_2 & 0 & \cdots & 0 \\ b_3 & 0 & d_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_n & 0 & \cdots & 0 & d_n \end{pmatrix}$ $(d_1 d_2 \cdots d_n \neq 0)$. Compute $\det(A)$.

5. **Determinant** $(3 \times 5 = 15 \text{ points})$

(a) Let $A, P \in \mathbb{R}^{n\times n}$ and suppose P is invertible. Prove that $\det(P^{-1}AP - \lambda I_n) = \det(A - \lambda I_n)$ for any $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

(b) Suppose $A \in \mathbb{R}^{n\times n}$ is an *orthogonal matrix*, i.e. $A^T A = I_n$. Prove that $\det(A) = 1$ or -1.

(c) Suppose A is a *skew-symmetric matrix* in $\mathbb{R}^{101\times 101}$, i.e. $A + A^T = 0$. Prove that $\det(A) = 0$.

6. **Linear Transformation** $(5 + 10 = 15 \text{ points})$

Let $b_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $b_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $b_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, and L be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined as $L(x) = x_1 b_1 + x_2 b_2 + (x_1 + x_2) b_3$.

(a) Show that L is a linear transformation.

(b) Find a matrix A such that $L(x) = Ax$ for all $x \in \mathbb{R}^2$.

7. **Bonus Problem** (10 points)

Set $\theta = \dfrac{2\pi}{n}$. A is a matrix of order n, $a_{s,t} = \sin(s+t)\theta, 1 \leq s,t \leq n$. Compute the determinant of $I_n + A$.