MAT2041 **TA: Meng ZHANG, Hang ZENG**

Assignment 6: Eigenvalues and eigenvectors

Assigned: Nov/27/23 Monday Due: Dec/4/23 Monday at 10:00pm

Rules (please read!!)

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. **Be sure to submit your homework as a single file.**
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you.

  However, only insightful discussions are allowed. Directly sharing the solutions is prohibited. The details of the collaboration policy for this course are available in the Resources tab on Piazza.

- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments!

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- **Total:** 100 points, 7 problems.

1. **Linear Transformation over $\mathbb{R}^2$** (20 points)

Set $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Consider the transformation that performs a vertical shear that maps $\boldsymbol{e}_1$ to $\boldsymbol{e}_1 + 2\boldsymbol{e}_2$ and leaves $\boldsymbol{e}_2$ unchanged. Determine matrix A that represents this transformation.

(b) Consider the transformation that rotates every vector counterclockwise by 45°. Determine matrix B that represents this transformation.

(c) Consider the transformation that first performs a vertical shear that maps $\boldsymbol{e}_1$ to $\boldsymbol{e}_1 + 2\boldsymbol{e}_2$ and leaves $\boldsymbol{e}_2$ unchanged and then rotates every vector counterclockwise by 45°. Determine matrix C that represents this transformation using matrix A and matrix B, and compute this matrix C.

2. **Compute Eigenpairs and Eigenvalues** (20 points)

Given the matrix

$$A = \begin{bmatrix} 2 & -4 & -4 \\ -4 & -1 & -1 \\ -4 & -1 & -1 \end{bmatrix}$$

(a) Compute the eigenvalues and corresponding eigenvectors of matrix A.

(b) Find an orthogonal matrix P s.t.

$$P^\top AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Here we further require $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

3. **Compute Eigenpairs and Eigenvalues** (20 points)

Given the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

(a) Find the eigenvalues of $AA^\top$.

(b) Find the eigenvalues and corresponding eigenvectors of $A^\top A$.

4. **Eigenvalues of Matrix Product** (10 points)

Let A and B be $n \times n$ matrices. Show that

(1) If λ is a nonzero eigenvalue of AB, then it is also an eigenvalue of BA.

(2) If $\lambda = 0$ is an eigenvalue of AB, then $\lambda = 0$ is also an eigenvalue of BA.

5. **Eigenvalues of Inverse Matrix** (10 points)

Suppose A is an invertible matrix.

(a) Suppose λ is an eigenvalue of A. Prove that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1};

(b) Suppose the multiset of eigenvalues of A is $\#\{\lambda_1, \ldots, \lambda_n\}$. Prove that the multiset of eigenvalues of A^{-1} is $\#\{\lambda_1^{-1}, \ldots, \lambda_n^{-1}\}$.

6. **Bounding a Quadratic Form** (10 points)

Suppose the eigenvalues of a real symmetric matrix A are $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ (allow repetition). Prove that $\lambda_1 \|x\|^2 \leq x^\top Ax \leq \lambda_n \|x\|^2$ for any $x \in \mathbb{R}^{n\times 1}$.

Hint: You may want to use a property on the orthogonal matrix.

7. **Eigenvalues** (10 points)

Suppose $\alpha, \beta \in \mathbb{R}$, $\alpha\beta \neq 0$. If $AB = \alpha A + \beta B$, Let λ be an eigenvalue of B, show that

(a) $\lambda \neq \alpha$.

(b) there is an eigenvalue μ of A, s.t.

$$\alpha\mu + \beta\lambda = \lambda\mu.$$