MAT2041 **TA: Meng ZHANG, Hang ZENG**

Assignment 7: Singular Value Decomposition and Quadratic Forms

Assigned: Dec/5/23 Tuesday Due: Dec 14/23 Thursday at 10:00pm

Rules (please read!!)

- **English:** Answer the questions in English. Otherwise, you'll lose half of the points.
- **Electronic submission:** Turn in solutions electronically via Blackboard. **Be sure to submit your homework as a single file.**
- **Collaboration policy:** Collaboration is allowed for all problems, but please list all the people with whom you discussed. Crediting help from other classmates will not take away any credit from you.

  However, only insightful discussions are allowed. Directly sharing the solutions is prohibited. The details of the collaboration policy for this course are available in the Resources tab on Piazza.

- **Questions on homework.** Start early and come to TA office hours with your questions on the assignments!

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- **Total:** 150 points, 6 problems.

1. **Judgement** (30 points)

State true or false. If true, provide brief reasons. If false, provide a counter-example.

(a) Any real square matrix A can be written as $A = SDS^T$ where D is a diagonal matrix, and S is an orthogonal matrix.

(b) If A and B are similar, then $det(A) = det(B)$.

(c) Any n by n real matrix A has at least one real eigenvalue.

(d) Any nonzero real matrix A has at least one nonzero singular value.

(e) For any real matrix A, $A^T A$ and AA^T have the same multiset of eigenvalues.

(f) If A and B have the same multset of eigenvalues, then A and B are similar.

2. **Full SVD** (30 points)

Find a full SVD (singular value decomposition) of each of the following matrices:

(a) $\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

3. **Minimum Norm Solution** (20 points)

Set A a matrix with full row rank r.

(a) Show that the full SVD of A can be written as

$$A = U \begin{bmatrix} D & 0 \end{bmatrix} V^T,$$

in which D is an invertible diagonal matrix of order r.

(b) Show that if $Ax = b$, then $\|x\| \geq \|D^{-1}U^T b\|$.

Hint: Write x as $x = \sum_{j=1}^{n} a_j v_j$ in $Ax = b$, and consider the first r entries of this sum. Here v_j is the j^{th} column of V.

4. **Pseudo-Inverse** (20 points)

Let $\boldsymbol{A}$ be $m \times n$ $(m \geq n)$ matrix with singular value decomposition $\boldsymbol{U}\Sigma\boldsymbol{V}^T$, where $\Sigma = \begin{bmatrix} D & 0 \end{bmatrix}$, D is a diagonal matrix with entries $\sigma_k, 1 \leq k \leq r$.

Suppose $\text{rank}(\boldsymbol{A}) = n$. Let Σ^+ denote the $n \times m$ matrix

$$\begin{pmatrix} \frac{1}{\sigma_1} & & & 0 & \dots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \frac{1}{\sigma_n} & 0 & \dots & 0 \end{pmatrix}$$

Define $\boldsymbol{A}^+ = \boldsymbol{V}\Sigma^+\boldsymbol{U}^T$.

(a) Show that

$$\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{I}_n.$$

(Note that $\boldsymbol{A}^+$ is called the *pseudo-inverse* of $\boldsymbol{A}$.)

(b) Show that $\hat{\boldsymbol{x}} = \boldsymbol{A}^+\boldsymbol{b}$ satisfies the normal equation $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$.

5. **Frobenius Norm** (20 points)

Suppose $\boldsymbol{A} = (A_{ij}) \in \mathbb{R}^{m \times n} (m \geq n)$ has an SVD (vector form of reduced SVD)

$$\boldsymbol{A} = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \cdots + \sigma_n \boldsymbol{u}_n \boldsymbol{v}_n^T$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Define $||\boldsymbol{A}||_F^2 = \sum_{i=1}^m \sum_{j=1}^n A_{ij}^2$ where A_{ij} is the (i, j)-th entry of A.

(a) Prove that $\|\boldsymbol{A}\|_F^2 = \mathrm{Tr}(A^T A)$ where $\mathrm{Tr}(\boldsymbol{B})$ denotes the trace of a matrix $\boldsymbol{B}$.

(b) Prove that $\|\boldsymbol{A}\|_F^2 = \sum_{i=1}^n \sigma_i^2$.

Hint for (b): You can use the relationship between trace and eigenvalues of a matrix or use the properties of orthogonal matrices.

6. **Rank-1 Decomposition** (30 points)

Suppose $\boldsymbol{x}_j \in \mathbb{R}^{m\times 1}, j = 1, \ldots, k$ are linearly independent, and $\boldsymbol{y}_j \in \mathbb{R}^{n\times 1}, j = 1, \ldots, k$ are linearly independent. Suppose $\boldsymbol{A} = \sum_{j=1}^{k} \boldsymbol{x}_j \boldsymbol{y}_j^\top$. In this problem, we will guide you to prove

$$\mathrm{C}(\boldsymbol{A}) = \mathrm{span}(\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\}).$$

(a) Prove the case of $k = 1$, i.e., if $A = \mathbf{u}\mathbf{v}^\top$ for nonzero vectors $\mathbf{u}, \mathbf{v}$, then $\mathrm{C}(\boldsymbol{A}) = \mathrm{span}(\{\mathbf{u}\})$.

(b) Prove $\mathrm{C}(\boldsymbol{A}) \subseteq \mathrm{span}(\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\})$.

Hint: There are multiple ways to prove. You can use the matrix form, or write columns of A in terms of $\mathbf{x}_j$'s and entries of $\mathbf{y}_j$'s.

(c) Prove $\mathrm{span}(\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\}) \subseteq \mathrm{C}(\boldsymbol{A})$.

Hint: One way to prove this is to use a matrix form, and solvability of a linear system with full row/column rank coefficient matrices in Lec 14.