Lecture 1

Vectors & Matrices

A Scalar is a real number, this is in contrast to vectors and matrices which may consist of many real numbers

Notation. Vectors in n-dimensional Space

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n, \quad v_i,\ i=1,\dots,n$$

are the components of v (or elements, or entries of v).

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{mn} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}$$

1. linear combinations of two vectors

i) $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Scalar multiplicasion $cv = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$

let $c=3$, $cv = 3\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

addition $v+w = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

ii) linear Combinasions, given two Scalar c, d.

$$cv + dw = \begin{pmatrix} 2c + 1d \\ 4c + 3d \end{pmatrix}$$

is a linear Combinasion of v and w with Coefficients c and d.

iii) let c, d be arbitrary reel numbers, can $cv+dw$ fill the whole xy-plane when $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The answer is yes

given $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, can we find c and d such that $cv + dw = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$?

More generally, let $A=[a_1, a_2, \cdots, a_n] \in \mathbb{R}^{m\times n}$ where a_i, $i=1,\cdots,n$ are the colums of A.

Define $Ax \equiv x_1 a_1 + \cdots + x_n a_n$, a linear combination of the columns of A. Here

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Two questions:

col(A), the colums space of A.

1) Is $\{Ax : x \in \mathbb{R}^n\} = \mathbb{R}^m$?

2) Given $b \in \mathbb{R}^m$, find x s.t. $Ax=b$.

2. Let's look at one example of Problem 2).

Given $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$, find c and d s.t.

$$c\begin{pmatrix} 2 \\ 1 \end{pmatrix} + d\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \quad \text{Here } A = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

The equivalent form in two equations

$$\begin{cases} 2c + 2d = 8 \\ c - d = 2 \end{cases}$$

Equation ② $- \frac{1}{2} \times$ Equation ① $\Rightarrow -2d = -2$

$\Rightarrow d = 1$ and $c = 3$. It's also easy to check

$$\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

More generally,

i) Column way: $c \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + d \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

ii) Row way: $\begin{cases} v_1 c + w_1 d = b_1 \\ v_2 c + w_2 d = b_2 \end{cases}$

iii) Matrix-vector way $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Actually $v_1 c + w_1 d$ is the dot-prod of $\begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$

Now let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $w = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$w = 3v$ In terms of equations

$$1c + 3d = 1$$
$$2c + 6d = 0$$

Equation 2 $-$ $2 \times$ Equation 1 $\Rightarrow$ $0 = -2$

No Solution!

However, let take $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $\begin{pmatrix} c \\ d \end{pmatrix}$ is a solution if $c = 1 - 3d$, so we have infinite number of solutions

In this case $\mathrm{col}(A) = \{ Ax : x \in \mathbb{R}^2 \}$

$= \{ c \begin{pmatrix} 1 \\ 2 \end{pmatrix} : c \in \mathbb{R} \}$ is a line

it does not fill the xy-plane!

3. $\mathbb{R}^3$. $v = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

v and w are not in the same line, but $cv + dw$ does not fill ~~the~~ $\mathbb{R}^3$!

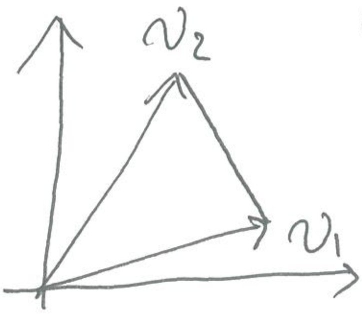
We need at least three vectors. here's one example $i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$ci + dj + ek = \begin{pmatrix} c \\ d \\ e \end{pmatrix}$ fill $\mathbb{R}^3$

In matrix-vector form $[i\ j\ k] \begin{bmatrix} c \\ d \\ e \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{identy mathix } I_3} \begin{bmatrix} c \\ d \\ e \end{bmatrix}$$

Problem. Given v_1 and v_2, describe the vectors on the line segment $\overline{v_1 v_2}$.



$cv_1 + dv_2$, where $c \geq 0$, $d \geq 0$, and $c + d = 1$.