Lecture 10 Matrix Inverses, Gauss-Jordan

1. **Definition** $A \in \mathbb{R}^{n\times n}$ $\underbrace{\text{has a inverse}}_{\text{invertible / nonsingular}}$ if there's $X \in \mathbb{R}^{n\times n}$ s.t. $AX = I_n$, $XA = I_n$.

2. Existence of Inverses

Th. If columns of $A \in \mathbb{R}^{n\times n}$ are independent, then A has inverse, which is also unique.

Proof. Uniqueness: let X_1, X be ~~also an~~ two inverse of A.

from $AX_1 = I_n$, $X(AX_1) = XI_n$, $\underbrace{(XA)}_{I_n}X_1 = X \Rightarrow X_1 = X$.

For existence, we first show for triangular matrices, they are invertible, iff. diagonal elements are nonzero.

It is easy to show for 2×2 case

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1} \end{pmatrix}$$

use induction, and the formula for block 2x2 case

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

From Gaussian elimination, if A has independent columns, then the LU decomposition A states

$$PA = LU$$

where P is a permutation matrix, and $P^{-1} = P^T$

L has all diagonals $= 1$, hence invertible

U has all diagonals nonzero, because A has independent columns, hence U is invertible.

Then $\quad A^{-1} = U^{-1}L^{-1}P.$

Group theoretic proof.

the existence of X s.t. $AX=I$ is guaranteed by Gaussian elimination.

Now we show X has independent columns

In fact, $Xx=0 \Rightarrow A\underbrace{Xx}_{=0}=x \Rightarrow x=0.$

Then by GE, there exists Y s.t. $XY=I$.

$A(XY)=AI. \Rightarrow \underbrace{(AX)}_{I}Y=A \Rightarrow Y=A.$ i.e. X is the inverse of A.

Another similar proof. We show A^T has independent columns $AX=I \Rightarrow X^TA^T=I$. $A^Tx=0 \Rightarrow$

$0=X^T\underbrace{A^T}x=x \Rightarrow x=0.$

By GE, there's Y, s.t. $A^TY=I$. or $Y^TA=I$.

$\Rightarrow Y^T=X$, i.e. X is the inverse of A.

when does A $\underbrace{\text{has an inverse}}_{\text{nonsingular}}$?

1) $(AB)^{-1} = B^{-1}A^{-1}$. (verify the definition)

2) A has indepent columns $\Longleftrightarrow$ A has inverse

$PA = LU$. L has 1 as its diagonals

U has nonzero diagonals.

P is permutation

3) Triangular matrices are nonsingular iff all its diagonals are nonzero.

~~Notes~~

$\Rightarrow$ The set of $n \times n$ nonsingular matrix forms a $\boxed{\text{group}}$ under ~~the~~ matrix multiplication, and the set of permutation matrices is a finite subgroup with cardinality $n!$

2.5 Computation of Inverses

$$AX = I = (e_1, e_2, \dots, e_n)$$

1) Compute the PLU decomposition of A

$$PA = LU$$

2) for $i = 1, \dots, n$. Compute the solution of

$$Ax_i = e_i$$

$$PLUx_i = e_i \qquad LUx_i = P^T e_i$$

i) $L c_i = P^T e_i$ ii) $U x_i = c_i$

3) $X = [x_1, \dots, x_n] \in R^{n \times n}$.

$\Rightarrow$ Notice that PLU decomposition is done once for each new rhs e_i, just two triangular solves!

3. Gauss-Jordan Elimination.

Augmented matrix (A, I_n), apply GE.

$$L^{-1}P(A, I_n) = (U, L^{-1}P)$$

where $PA = LU$. now we use GE to reduce U to I_n, we have

$$U^{-1}(U, L^{-1}P) = (I_n, U^{-1}L^{-1}P)$$

$$= (I_n, A^{-1})$$

Again we use block matrices

$U = \left(\begin{array}{c|c} U_1 & \hat{u}_1 \\ \hline 0 & u_{nn} \end{array}\right)$, it's easy to verify

$$\Rightarrow \left(\begin{array}{c|c} I_{n-1} & -\hat{u}_1 u_{nn}^{-1} \\ \hline 0 & u_{nn}^{-1} \end{array}\right) U = \left(\begin{array}{c|c} U_1 & 0 \\ \hline 0 & 1 \end{array}\right) \Rightarrow \text{block-matrix format.}$$

$$\text{let } U_1 = \left(\begin{array}{c|c} U_2 & u_2 \\ \hline 0 & u_{n-1,n-1} \end{array}\right) \in R^{(n-1)\times(n-1)}$$

$$\left(\begin{array}{c|c|c} I_{n-2} & -u_2 u_{n-1,n-1}^{-1} & \\ \hline 0 & u_{n-1,n-1}^{-1} & \\ \hline 0 & & 1 \end{array}\right)\left(\begin{array}{c|c} I_{n-1} & -u_1 u_{nn}^{-1} \\ \hline & u_{nn}^{-1} \end{array}\right) U$$

$$= \left(\begin{array}{c|c|c} U_2 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & & 1 \end{array}\right)$$

It is easy to see $U^{-1} =$.

$$\left(\begin{array}{c|c} u_{11}^{-1} & \\ \hline & I_{n-1} \end{array}\right)\left(\begin{array}{c|c|c} 1 & -u_{12} u_{22}^{-1} & \\ \hline 0 & u_{22}^{-1} & \\ \hline & & I_{n-2} \end{array}\right) \cdots \left(\begin{array}{c|c|c} I_{n-2} & -u_2 u_{n-1,n-1}^{-1} & \\ \hline & u_{n-1,n-1}^{-1} & \\ \hline & & 1 \end{array}\right)\left(\begin{array}{c|c} I_{n-1} & -u_1 u_{nn}^{-1} \\ \hline 0 & u_{nn}^{-1} \end{array}\right)$$

<u>Example</u>.

$$\left(\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{array}\;\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{ccc} 11 & -4 & -3 \\ -4 & 2 & 1 \\ -3 & 1 & 1 \end{array}\Bigg)$$

let $I_n = (e_1, e_2, \cdots, e_n) = \begin{pmatrix} e_1^T \\ \vdots \\ e_n^T \end{pmatrix} \in \mathbb{R}^{n\times n}$

$\mathcal{U} \in \mathbb{R}^{n\times n}$ upper triangular, $u_{ii} \neq 0$. diag entries $\neq 0$.

$U = \mathcal{U} I_n = [\overset{\swarrow \text{ Notation!}}{u_1}, \cdots, u_n] \begin{pmatrix} e_1^T \\ \vdots \\ e_n^T \end{pmatrix} = u_1 e_1^T + \cdots + u_n e_n^T$

$I_n = e_1 e_1^T + \cdots + e_n e_n^T$

$\& \; \mathcal{U} = I + (u_1 - e_1) e_1^T + \cdots + (u_n - e_n)^T e_n,$

$= I + \hat{u}_1 e_1^T + \cdots + \hat{u}_n e_n^T$ let $\hat{u}_i = u_i - e_i$

$= (I + \hat{u}_n e_n^T)(I + \hat{u}_{n-1} e_{n-1}^T) \cdots (I + \hat{u}_1 e_1^T)$

Notice that $(I + \hat{u}_i e_i^T)^{-1} = I - \frac{1}{u_{ii}} \hat{u}_i e_i^T$

Use Sherman-Morrison:

Then: $\mathcal{U}^{-1} = \left(I - \frac{\hat{u}_1}{u_{11}} e_1^T\right) \cdots \left(I - \frac{\hat{u}_{n-1}}{u_{n-1,n-1}} e_{n-1}^T\right)\left(I - \frac{\hat{u}_n}{u_{nn}} e_n^T\right)$

4. Computational cost of LU and matrix inversion

$$\left(\begin{array}{c|c} 1 & \\ \hline -l_1 & I_{n-1} \end{array}\right)\left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline a_1 & A_1 \end{array}\right) = \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & A_1 - l_1 u_1^T \end{array}\right)$$

$l_1 = a_1 / a_{11}$ $(n-1)$ divisions

$l_1 u_1^T$ $(n-1)^2$ multiplications

$A_1 - l_1 u_1^T$ $(n-1)^2$ subtractions

$n(n-1)$ "$*/\div$", $(n-1)^2$ "$-$"

$n^2 + (n-1)^2 + \cdots + 1^2 \approx \frac{1}{3}n^3 \Rightarrow \frac{2}{3}n^3$ "ops"

triangular solve

$$\left(\begin{array}{c|c} U_1 & u_1 \\ \hline 0 & u_{nn} \end{array}\right)\left(\begin{array}{c} \tilde{x} \\ \hline x_n \end{array}\right) = \left(\begin{array}{c} \tilde{c} \\ \hline c_n \end{array}\right)$$

$x_n = c_n / u_{nn}$

$U_1 \tilde{x} = \underbrace{-u_1 x_n + \tilde{c}}_{n-1 \text{ "}*\text{"},\ n-1 \text{ "}-\text{"}}$

$n + n-1 + \cdots + 1 \approx \frac{1}{2}n^2 = n^2.$

Matrix inversion $AX = I_n$.

i) LU. $\frac{2}{3}n^3$

ii) n triangular solve $n \cdot n^2 = n^3$

$\frac{2}{3}n^3 + n^3 = \frac{5}{2}n^3$ "op"

5. $(A^T)^{-1} = (A^{-1})^T$, easy to check $(A^{-1})^T A^T$

$$= (A \cdot A^{-1})^T = I_n^T = I_n.$$

6. Inner-prod. $u, v \in \mathbb{R}^{n\times 1}$

$$u \cdot v = \bar{u}^T v \qquad (Ax)^T y = x^T (A^T y)$$

7. Symmetric matrices

$$S \text{ is symmetric iff. } S^T = S$$

The inverse of invertible symmetric matrix is symetric: $(S^{-1})^T = (S^T)^{-1} = S^{-1}$

$\forall A \in \mathbb{R}^{m\times n}$, $A^TA \in \mathbb{R}^{n\times n}$, $AA^T \in \mathbb{R}^{m\times m}$ symmetric

$S = \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}$, $S = S^T$ block elimination $\Rightarrow \begin{pmatrix} 2 & A \\ 0 & -A^TA \end{pmatrix}$

S invertible iff. A^TA invertible iff A has full col rank