Lecture 11, Vector Spaces and general linear systems.

We consider $Ax=b$, for $A \in R^{m \times n}$. We first consider $Ax=0$. We know if columns of A independent $\Rightarrow x=0$. If not, $Ax=0$ has nonzero solutions.

$\forall x, y$, s.t. $Ax=0$, $Ay=0$. we have

$$A(\alpha x+\beta y) = \alpha Ax + \beta Ay = 0.$$ i.e., $\alpha x+\beta y$ is also a solution

We say the solution set of $Ax=0$, $N(A)$, the null space of A is a vector space.

$$N(A) = \{ x \in R^n, Ax=0 \}$$

e.g. a line passing through $(0,0)$ on R^2 is a vector space

$(x,y) \neq (0,0)$ is not.

A vector space is a space with $x+y$, αx defined satisfying 8 rules. Examples, 1) $R^{m\times n}$, 2) polynomials of degree $\leq n$.

A subspace S of a vector space V.

if $\forall \alpha, \beta \in R$, $x, y \in S$. $\alpha x + \beta y \in S$.

Examples: $C(A)$ column space of $A \in R^{m\times n}$ is a subspace of R^m, $N(A)$ is a subspace of R^n.

$Ax=b$ has a solution iff. $b \in C(A)$ iff. $b \in \text{Span}\{a_1, a_2, \ldots, a_n\}$, $A = (a_1, a_2, \ldots, a_n)$

The row space of A is $C(A^T)$.

$\text{rank}(A) =$ max # of independent columns of A

$=$ " " " rows of A.

Intersection of two subspaces.

S, T are two subspaces, then $S \cap T$ is also a subspace, E.g. S is a line going through the origin in $\mathbb{R}^3$, and T a plane going through the origin in $\mathbb{R}^3$. Then $S \cap T$ can be

1) $\{0\}$, 2) S, if $S \subset T$.

Sum of two subspaces

$$S+T = \{x+y : x \in S, y \in T\}$$

e.g. $S = C(A)$, $T = C(B)$. $S+T = C([A, B])$

Direct sum: if $S \cap T = \{0\}$ $S+T = S \oplus T$.

$$\dim(S) + \dim(T) = \dim(S+T) + \dim(S \cap T)$$

So $\dim(S) + \dim(T) = \dim(S \oplus T)$.

A set of vectors $v_1, \dots, v_s$ independent if.

$$\alpha_1 v_1 + \dots + \alpha_s v_s = 0 \Rightarrow \alpha_1 = \dots = \alpha_s = 0.$$

$\dim(S)$ = max # of independent vectors of S.

E.g. the set of 2×2 matrices $:= M_2$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

are independent $\Rightarrow \dim(M_2) = 4.$

We consider $Ax=0$, $A \in \mathbb{R}^{m\times n}$

when A $\in \mathbb{R}^{n\times n}$ is square and nonsingular, GE with (partial) pivoting allows to find n (nonzero) pivots which are the diagonal elements of U in

$$PA = LU$$

P: permutation, L: unit lower tri, U: upper tri

When $A \in \mathbb{R}^{n\times n}$ is singular, we will not be able to find n pivots, ∴ for $A \in \mathbb{R}^{m\times n}$, the number of pivots we can find is $r = \operatorname{rank}(A)$.

Take a rectangular $A \in \mathbb{R}^{m\times n}$, assume up to $k-1$ steps, we can apply GE with pivoting

$$E_{k-1}P_{k-1}\cdots E_1P_1A = \begin{array}{cc} & \begin{array}{cc} k-1 & \end{array} \\ k-1 & \left(\begin{array}{c|c} * & * \\ \hline 0 & A_k \end{array}\right) \end{array},$$

where "$*$" indicate a possible nonzero ~~bloc~~ block/matrix.

$\Rightarrow$ P_i's are row exchanges

$\Longrightarrow$ which may change from step to step

However, the first (few) columns of A_k is (are) zeros. Write

$$A_k = \left(0 \,\middle|\, a_k, * \right)$$

where a_k is the first nonzero col of A_k. (If $A_k = 0$, the process is terminated). Then use permutation $\widetilde{P}_k$, s.t.

$$\widetilde{P}_k A_k = \left(0 \,\middle|\, \begin{array}{c|c} \alpha_k & * \\ \hline * & * \end{array} \right), \quad \alpha_k \neq 0$$

Use elimination matrix $\widetilde{E}_k$, s.t.

$$\widetilde{E}_k \widetilde{P}_k A_k = \left(0 \,\middle|\, \begin{array}{c|c} \alpha_k & * \\ \hline 0 & A_{k+1} \end{array} \right)$$

If $A_{k+1} = 0$, we terminate, otherwise, we apply the same procedure to A_{k+1}.

As we discussed in GE with pivoting,

$$E_r P_r \cdots E_1 P_1 = \hat{E}_r \hat{E}_{r-1} \cdots \hat{E}_1 P_r P_{r-1} \cdots P_1.$$

Then

$$P_r P_{r-1} \cdots P_1 A = \hat{E}_r^{-1} \cdots \hat{E}_1^{-1} \mathcal{U}_0.$$

$$\hat{E}_r^{-1} \cdots \hat{E}_1^{-1} = \left(\begin{array}{ccc|c} 1 & & & \\ * & \ddots & & 0 \\ \vdots & & 1 & \\ \hline \vdots & & * & \\ \vdots & & \vdots & I_{m-r} \\ * & \cdots & x & \end{array}\right) \equiv \left(\begin{array}{c|c} L_1 & 0 \\ \hline L_2 & I_{m-r} \end{array}\right)$$

Now let $Q_1 = (1, n_1)$ exchange of col 1 and n_1, $\ldots$, $Q_r = (r, n_r)$, then

$$\underbrace{P_r \cdots P_1}_{P} A \underbrace{Q_1 \cdots Q_r}_{Q} = \left(\begin{array}{c|c} L_1 & 0 \\ \hline L_2 & I_{m-r} \end{array}\right) \left(\begin{array}{ccc|c} \alpha_1 & & * & \\ & \ddots & & * \\ & & \alpha_r & \\ \hline & 0 & & \underbrace{0}_{n-r} \end{array}\right)$$

$$= \left(\begin{array}{c|c} L_1 & 0 \\ \hline L_2 & I_{m-r} \end{array}\right) \left(\begin{array}{c|c} U_1 & U_2 \\ \hline 0 & 0 \end{array}\right)$$

$$= \overset{r}{\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}} (U_1, U_2)_r$$

$$\cancel{\begin{pmatrix} I & \\ & E_k \end{pmatrix}\begin{pmatrix} I & \\ & P_k \end{pmatrix} E_{k-1}P_{k-1}\cdots E_1P_1A = \left(\begin{array}{c|cc} * & \multicolumn{2}{c}{*} \\ \hline 0 & \alpha_k & * \\ \cline{2-3} & 0 & A_{k+1} \end{array}\right)}$$

$$= \left(\begin{array}{c|c|c|c} * & \multicolumn{3}{c}{*} \\ \hline 0 & 0 & \alpha_k & * \\ \cline{3-4} & & 0 & A_{k+1} \end{array}\right)$$

$$\begin{array}{cccc} & & \uparrow & \\ & & n_k & \end{array}$$

n_k the index of the col corresponding to α_k

After r steps, we have

$$\begin{array}{c} \begin{array}{ccccccc} & n_1 & & n_2 & & n_k & \end{array} \\ E_rP_r\cdots E_1P_1A = \left(\begin{array}{ccccccccc} 0 & \alpha_1 & * & \cdots & & & & & * \\ \hline 0 & \cdots & 0 & \alpha_2 & * & \cdots & & & * \\ \hline & & & \vdots & & & & & \\ \hline 0 & & \cdots & & & 0 & \alpha_r & * & \cdots * \\ \hline & & & & 0 & & & & \end{array}\right) = U_0 \end{array}$$

U_0 is called a row echelon form (REF) of A.

The pivots, α_k, rather than always on the main diagonal in the nonsingular case, are possibly pushing towards the <u>right</u> of the matrix.

Reduced row echelon form (RREF)

Further apply Gauss–Jordan to

$$(U_1, U_2) \longrightarrow (I_r, U_1^{-1}U_2)$$

$$PAQ = \begin{pmatrix} L_1U_1 & 0 \\ L_2U_1 & I \end{pmatrix} \begin{pmatrix} I_r, & \overbrace{U_1^{-1}U_2}^{F} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} I_r & \overbrace{U_1^{-1}U_2}^{F} \\ 0 & 0 \end{pmatrix} Q^T \equiv R_0$$ reduced row echelon form RREF.

$$= \begin{array}{c} \begin{array}{ccccccc} & n_1 & & n_2 & & n_r & \end{array} \\ \left(\begin{array}{cccccccc} 0 \cdots 0 & 1 & * & 0 & * & 0 & \cdots & * \\ & & & 1 & * & 0 & \cdots & \\ & & & & & \vdots & & \\ & & & & & 0 & \cdots & \\ & & & & & 1 & * \cdots & * \\ \hline & & & & O & & & \end{array}\right) \end{array} \begin{array}{l} r \\ \\ m-r \end{array}$$

$$A = P^T \underbrace{\begin{pmatrix} L_1U_1 \\ L_2U_1 \end{pmatrix}}_{C} R = CR. \quad C = (a_{n_1}, \dots, a_{n_r})$$

where $A = (a_1, \dots, a_r)$

~~too~~

Two important special cases.

1). $A \in \mathbb{R}^{m\times n}$ is full column rank.

$r = n \leqslant m$, A is tall matrix.

$$PA = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} U_1 . \quad L_1 \in \mathbb{R}^{n\times n}, \text{ lower tri.} \quad U_1 \in \mathbb{R}^{n\times n}, \text{ upper tri.}$$

$$= \begin{pmatrix} L_1 U_1 \\ L_2 U_2 \end{pmatrix} I_n .$$

$C = A$, $R = I_n$ ~~redu~~ RREF

2) $A \in \mathbb{R}^{m\times n}$ full row rank

$r = m \leqslant n$. A is fat matrix.

$$A Q = L_1 (U_1, U_2) \quad L_1 \in \mathbb{R}^{m\times m} \text{ lower tri}$$

$$= L_1 U_1 (I_m, U_1^{-1} U_2) \quad U_1 \in \mathbb{R}^{m\times n} \quad \text{upper tri.}$$

$C = L_1 U_1$, $R = (I_m, U_1^{-1} U_2)$.

Now we show $\operatorname{rank}(A) = r$. # of pivots.

$$C = P^T \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} U_1.$$

L_1 is unit lower tri

L_1 is invertible

$$U_1 = \begin{pmatrix} \alpha_1 & & * \\ & \alpha_2 & \\ 0 & & \ddots \ \alpha_r \end{pmatrix}$$

is invertible $\Rightarrow$ C has r independent col's.

$A = C(I_r, F)$, any col of is a linear comb of col's of C

For any s set of columns of A. $s > r$

$$(a_{i_1}, \cdots, a_{i_s}) = CG, \quad G \in \mathbb{R}^{r \times s}$$

Since $s > r$, there's nonzero $g \in \mathbb{R}^s$, $Gg = 0$.

$(a_{i_1}, \cdots, a_{i_s})\, g = 0. \Rightarrow a_{i_1}, \cdots, a_{i_s}$ independent

Col rank of $A = r$.

Now $A^T = \begin{pmatrix} I_r \\ F^T \end{pmatrix} C^T = \begin{pmatrix} I_r \\ F^T \end{pmatrix} \left(U_1^T L_1^T,\ U_1^T L_2^T \right) P$

$$= \begin{pmatrix} U_1^T L_1^T \\ F^T U_1^T L_1^T \end{pmatrix} \left(I_r,\ \cancel{(U_1^T L_1^T)}\ L_1^{-T} L_2^T \right) P$$

$\Rightarrow$ row rank of $A = r$

Block elimination.

$$PAQ = \begin{array}{cc} & \begin{array}{cc} r & \end{array} \\ r & \begin{pmatrix} W & H \\ J & K \end{pmatrix} \end{array}, \qquad W \text{ is invertible and } r(A)=r.$$

$$\to \begin{pmatrix} W & H \\ 0 & K - JW^{-1}H \end{pmatrix} = \begin{pmatrix} W & H \\ 0 & 0 \end{pmatrix} \text{ if } r(A)=r.$$

$$\to \begin{pmatrix} I_r & W^{-1}H \\ 0 & 0 \end{pmatrix}$$

$$PAQ = \begin{pmatrix} W \\ J \end{pmatrix} W^{-1} (W, H) = CW^{-1}B$$