Lecture 12 Solving general $Ax=b$

Given $A \in R^{m\times n}$, $b \in R^m$ if $Ax=b$ has no sol, the solution set of $Ax=b$ is ϕ: the empty set

When A is square and nonsingular, $Ax=b$ has an unique solution $x = A^{-1}b$.

When the solution is not unique, say x_0, y_0 both are solutions: $Ax_0 = b$, $Ay_0 = b \Rightarrow$

$A(x_0 - y_0) = 0$. $\Rightarrow x_0 - y_0 \in N(A)$, the nullspace of A.

i.e., $\forall g \in N(A)$ $x = y_0 + g$ is a solution, when $Ay_0 = b$: the general sol is the sum of a specific solution y plus a ~~null~~ ~~solution~~ vector g, satisfying $Ag=0$

Any specific sol will do: $x = x_0 + g$. is also a general solution

In general,
~~Or~~ we write $x \in x_0 + N(A)$.

Sol set of $Ax=b$, if ~~exists~~ $Ax=b$ has a solution, can be written as

$$x_0 + N(A) = \{x_0 + g : g \in N(A), Ax_0 = b\}$$

Such a set is <u>not</u> a linear subspace when $b \neq 0$, but rather an <u>affine</u> subspace. It's easy to verify

$\forall \alpha \in R$, $x \in x_0 + N(A)$, $y \in x_0 + N(A)$ implies

$$\underbrace{\alpha x + (1-\alpha) y}_{\text{an affine combination (rather than a linear combination)}} \in x_0 + N(A)$$

Solution set. $S(A,b) = \{x \in R^n : Ax = b\}$ is a geometric object, which can be described in two different ways. Solving $Ax=b$ amounts to go from the <u>implicit</u> description to the <u>explicit</u> description

Solution set of $Ax=b$, $A\in R^{m\times n}$.

1) <u>implicit</u> description, as

$$\{x\in R^n : Ax=b\}$$

for example

$$\{x\in R^2 : 2x_1+3x_2=1\}$$

is a line in R^2. (not throug the origin $\begin{pmatrix}0\\0\end{pmatrix}$)

2) <u>explicit</u> description (parametric description)

$$\left\{x\in R^2 : \begin{cases} x_1=(1-3t)/2 \\ x_2=t \end{cases}, t\in R\right\}$$

Compare with the description / representation of unit cycle in R^2

1) Implicit: $\{x\in R^2 : x_1^2+y_1^2=1\}$

2) Explicit: $\left\{x\in R^2 : \begin{cases} x_1=\cos\theta, \\ y_1=\sin\theta, \end{cases} 0\le\theta\le 2\pi\right\}$

Applying RREF:

Solving $Ax=0. \Leftrightarrow \underbrace{PAQ}\,Q^Tx=0.$ $\quad Q^Tx=\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} I_r & \overset{F}{\cancel{A_1^{-1}A_2}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0.$$

$$x_1+Fx_2=0. \qquad x_1=-Fx_2$$

$$Q^Tx=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\cancel{Q}\begin{pmatrix}-F\\ I_{n-r}\end{pmatrix}x_2.\ ,\forall x_2\in R^{n-r}$$

$$x=Q\begin{pmatrix}-F\\ I_{n-r}\end{pmatrix}x_2. \Rightarrow N(A)=\underline{\underline{\text{Span}}}\left(Q\begin{pmatrix}-F\\ I_{n-r}\end{pmatrix}\right)$$

$$=C\left(Q\begin{pmatrix}-F\\ I_{n-r}\end{pmatrix}\right)$$

<u>Examples</u> 1) $A=\begin{pmatrix}1&2&11&17\\3&7&35&57\end{pmatrix}\to\begin{pmatrix}1&2&11&17\\0&1&4&6\end{pmatrix}$

$$\to\begin{pmatrix}\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I_2}&\underbrace{\begin{matrix}3&5\\4&6\end{matrix}}_{F.}\end{pmatrix}\equiv \underline{\underline{R}} \qquad Q=I_4.$$

$$x=\begin{pmatrix}-3&-5\\-4&-6\\1&0\\0&1\end{pmatrix}\begin{pmatrix}x_3\\x_4\end{pmatrix},\ \text{where}\begin{pmatrix}x_3\\x_4\end{pmatrix}\in R^2.$$

(free variables)

In terms of $\underline{\underline{\text{CR}}}$ decomposition

$\underline{\underline{C}}=\begin{pmatrix}1&2\\3&7\end{pmatrix}$, $\quad \boxed{A=CR}$

2) $A=\begin{pmatrix}1&7&3&35\\2&14&6&70\\2&14&9&97\end{pmatrix}\to\begin{pmatrix}1&7&3&35\\0&0&0&0\\0&\underset{\substack{\uparrow\\ \text{zero col.}}}{0}&3&27\end{pmatrix}$

$\to\begin{pmatrix}1&7&3&35\\0&0&3&27\\0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&7&3&35\\0&0&1&9\\0&0&0&0\end{pmatrix}\to$

$\begin{pmatrix}1&7&0&8\\0&0&1&9\\\underset{\substack{\uparrow\\ n_1=1}}{0}&0&\underset{\substack{\uparrow\\ n_2=3}}{0}&0\end{pmatrix}\equiv R_0.\quad \underline{\underline{R}}=\begin{pmatrix}1&7&0&8\\0&0&1&9\end{pmatrix}$

$Q_1=(1),\quad Q_2=(2,3),\quad Q=Q_1Q_2=Q_2.$

$RQ=\begin{pmatrix}1&0&7&8\\0&1&\underbrace{0\;\;9}_{F}\end{pmatrix}$

$Q^Tx=\begin{pmatrix}-7&-8\\0&-9\\1&0\\0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix},\quad \begin{pmatrix}y_1\\y_2\end{pmatrix}\in\mathbb{R}^2.$

$x=\begin{pmatrix}-7&-8\\1&0\\0&-9\\0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}.$

$\Rightarrow C=\begin{pmatrix}1&3\\2&6\\\underset{a_1}{2}&\underset{a_3}{9}\end{pmatrix},\quad A=CR.$

For general $Ax=b$.

Find RREF for $[A, b] \to \left(\begin{array}{cc|c} I_r & F & d_1 \\ \hline 0 & 0 & d_2 \end{array}\right)$

$Ax=b$ has a solution iff $d_2=0$. $Q^T x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$(I_r, F)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cancel{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}}\, d_1 .$$

a specific sol: set $x_2=0$. $x_1=d_1$.

$$x_0 = Q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q\begin{pmatrix} d_1 \\ 0 \end{pmatrix}$$

On the other hand, the ~~the~~ general sol of $Ax=0$

$$x = Q\begin{pmatrix} -F \\ I_{n-r} \end{pmatrix} y, \quad y \in R^{n-r}$$

The general sol for $Ax=b$, $\boxed{\text{when } d_2=0,}$ is

$$x = Q\left(\begin{pmatrix} d_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -F \\ I_{n-r} \end{pmatrix} y\right), \quad \forall y \in R^{n-r}$$

otherwise, when $d_2 \neq 0$. the solution set of $Ax=b$ is ϕ.

Example

$$[A, b] = \begin{pmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{pmatrix} \to$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ \hline 0 & 0 & 0 & 0 & b_3-b_1-b_2 \end{array}\right) = (R_0, \hat{b}).$$

$Ax=b$ has sol iff $b_3-b_1-b_2=0$.

Use $Q=(2,3)$. $Q^T x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 3 & 2 & b_1 \\ 0 & 1 & 0 & 4 & b_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$ a Specific sol:

Setting $y_3=0$, $y_4=0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$Q^T x = y = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \quad \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \in \mathbb{R}^2$$

$$x = \begin{pmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_3 \\ y_4 \end{pmatrix},$$

Example $A=\begin{pmatrix}1&1\\1&2\\-2&-3\end{pmatrix}$, $b=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

Augmented matrix

$$\begin{pmatrix}1&1&b_1\\1&2&b_2\\-2&-3&b_3\end{pmatrix}\rightarrow\begin{pmatrix}1&1&b_1\\0&1&b_2-b_1\\0&-1&b_3+2b_1\end{pmatrix}\rightarrow\begin{pmatrix}1&0&2b_1-b_2\\0&1&b_2-b_1\\0&0&b_3+b_1+b_2\end{pmatrix}$$

$Ax=b$ has a sol iff. $b_3+b_1+b_2=0$.

A has full col rank, $r=n=2$. If $b_1+b_2+b_3=0$.

Sol is unique $\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}2b_1-b_2\\b_2-b_1\end{pmatrix}$

RREF: $R_0=\begin{pmatrix}I\\0\end{pmatrix}$, $R=I$

<u>Sumary</u>. R_0: $[I]$ $[I,F]$ $\begin{bmatrix}I\\0\end{bmatrix}$ $\begin{bmatrix}I&F\\0&0\end{bmatrix}$

$A\in\mathbb{R}^{m\times n}$ $r=m=n$ $r=m<n$ $r=n<m$ $r<m, r<n$.