lecture 13 Independence, rank, ~~basis, dimension~~

Def. A set of vectors $v_1, \cdots v_n$ is independent if none of the vectors can be written as LC of other vectors

(LC: linear combination) $\forall v_i, v_i \notin \text{Span}\{v_1, \cdots v_{i-1}, v_{i+1}, \cdots v_n\}$

$\Rightarrow$ $v_1 \cdots v_n$ are dependent if one of the vectors can be written as LC of other vectors

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<u>Theorem</u> $v_1, \cdots, v_n$ independent iff. $\underbrace{[v_1, \cdots v_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{LC} = 0$

$\Rightarrow x_1 = \cdots = x_n = 0$. $\left(\Leftrightarrow \text{NullSpace}([v_1, \cdots, v_n]) = \{0\}. \right)$

<u>Proof</u>. The theorem is equivalent to the following.

$v_1 \cdots, v_n$ dependent iff there's nonzero x

s.t. $[v_1, \cdots v_n] x = 0$. $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

let's prove this equivalent form:

"$\Rightarrow$" <u>w.o.l.g.</u> v_1 is a linear combination of

$v_2, v_3, \cdots, v_n$

$$v_1 = \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$-v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \Rightarrow [v_1, v_2, \cdots v_n] \begin{bmatrix} -1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = 0$$

$$x = \begin{pmatrix} -1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \neq 0.$$

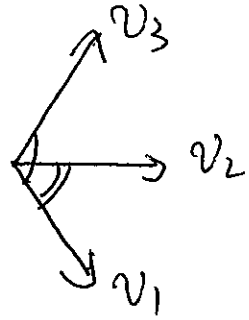
"$\Leftarrow$" let $x \neq 0$. and $[v_1, \cdots, v_n]x = 0$.

w.o.l.g. $x_1 \neq 0$. $x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$

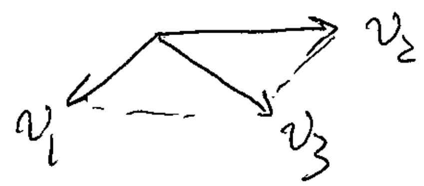
$$v_1 = \left(-\frac{x_2}{x_1}\right) v_2 + \cdots + \left(-\frac{x_n}{x_1}\right) v_n.$$

We can say there's no extra vectors in an independent set because none ~~to~~ can be written as LC of other vectors; each vector in the set needs to contribute something new.

Geometrically,



independent.



dependent.

Some examples: 1) $0, v_2, \dots v_n$ dependent.

2) $\underline{v_1, 2v_1}, v_3, \dots v_n$ dependent.

3) Columns of $I_n = (e_1, \dots, e_n)$ independent.

4) col's of triangular matrices independent iff diagonal elements are nonzero

5) $A \in R^{n \times n}$, col's of A independent iff A invertible, i.e. there's X s.t. $XA = I$, $AX = I$.

6) Col's of A dependent $\Rightarrow$ col's of $[A, B]$ dependent.

7) col's of A independent $\Rightarrow$ col's of $\begin{pmatrix} A \\ B \end{pmatrix}$ independent.
$\Rightarrow$ a subset of col's of A is also independent

8) $A \in R^{n \times n}$, A has independent col's iff $Ax = b$ has unique solution $\forall b \in R^n$.

9) $A \in R^{n \times n}$, A has independent col's iff A^T has independent col's. (proof. "$\Rightarrow$" If u is s.t. $A^T u = 0$, we need to show $u = 0$. A col's independent, $\underbrace{\exists x, Ax = u}$

$A^T A x = 0 \Rightarrow u^T A^T A x = 0$. $(Ax)^T A x = 0 \Rightarrow u = Ax = 0$.

($A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$
A, B full col rank $\Rightarrow AB$ full col rank)

$A \in \mathbb{R}^{m\times n}$, if A has independent col's, we say

A has full column rank. In this case: $m \geq n$.

and $Ax=0 \Rightarrow x=0$.

10) In fact, if $m<n$, ~~A has dependent col's.~~

col's of A are dependent. (Such matrix is called a fat matrix)

Proof. take the first m col's of A, call it $A_1 \in \mathbb{R}^{m\times m}$

~~1) if A_1 has independent col's. Consider~~

$$[A_1, A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \qquad x_1 = -A_1^{-1}A_2 x_2$$

Select an arbitrary $x_2 \neq 0$, compute x_1 as above

then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$, and $Ax=0 \Rightarrow$ col's A dependent.

2) if A_1 has dependent col's, then of cause ~~A~~ $[A_1, A_2]$ has dependent col's. #

There's another approach using induction on m, and one-step of GE

11). If $A \in \mathbb{R}^{m\times m}$ is invertible, and $v_1, \cdots, v_m$ are independent, then $Av_1, \cdots, Av_m$ are also independent

proof. Consider, $[Av_1, \cdots, Av_n]x = 0.$

$$A[v_1, \cdots, v_n]x = 0. \Rightarrow [v_1, \cdots v_n]x = 0 \Rightarrow x = 0.$$

12) Independence and RREF (or REF): Pivot col's of RREF of $A \in \mathbb{R}^{m\times n}$ gives the $\underbrace{\text{first } r}$ independent cols of A: r is the rank of A, $r = \operatorname{rank}(A)$ (col rank = row rank)

<u>rank</u> which is defined to be the maximal number of ~~inde~~ independent col's of A.

$$\begin{bmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & & 1 & * \cdots * & 0 & * \cdots * \\ & & & & \ddots & \vdots & \\ & & & & & \underline{1} & * \cdots * \\ \hline 0 & & \cdots & & & & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}$$

$$\begin{array}{ccccc} \Downarrow & & \Downarrow & & \Downarrow \\ n_1 & & n_2 & \cdots & n_r \end{array}$$

$A = [a_1, \cdots, a_n]$

$\{a_{n_1}, a_{n_2}, \cdots a_{n_r}\}$ are independent!

rank of a set of vectors

$S=\{v_1,\dots,v_n\}$

Form the matrix $V=(v_1,\dots,v_n)$,

the rank of $S = \operatorname{rank}(V)$. This is the same as the # of vectors in ~~the~~ the maximal linearly independent systems of S:

$$\{v_{r_1},\dots v_{r_k}\}$$ linearly independent,

but adding another vector v_i into the above, they become dependent.

The computation of REF/RREF of $V=(v_1,\dots,v_n)$ can find the maximal linearly independent system.

Properties of matrix rank:

13) $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, then

$$\operatorname{rank}(AB) \leq \operatorname{rank}(A), \quad \operatorname{rank}(AB) \leq \operatorname{rank}(B)$$

Proof. we first prove the 2nd one. use the following fact: if $v_1, \dots, v_n$ are dependent, then

$$Av_1, \dots, Av_n \text{ are also dependent}$$

for any matrix A s.t. Av_1 is defined.

Now let $B = (b_1, \dots, b_n)$, then for any subset of col's of B, say $b_{i_1}, \dots, b_{i_k}$, if $k > \operatorname{rank}(B)$ this set of col's is dependent. $\Rightarrow$

$$Ab_{i_1}, \dots Ab_{i_k} \text{ are dependent.}$$

$\Rightarrow$ Any k col's of AB, if $k > \operatorname{rank}(B)$, are dependent.

hence $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$

Now $\operatorname{rank}(AB) = \operatorname{rank}((AB)^T) = \operatorname{rank}(B^T A^T)$

$\leq \operatorname{rank}(A^T) = \operatorname{rank}(A)$

14) $\operatorname{rank}(A) \leq \operatorname{rank}([A, B])$

<u>proof</u>. The maximal # of independent col's can only increase by adding more vectors to the set.

15) $Ax=b$ has a solution iff $\operatorname{rank}(A)=\operatorname{rank}([A,b])$

The coefficient matrix A and the augmented matrix $[A, b]$ have the same rank

<u>Proof</u>. "$\Rightarrow$" $r(A) \leq r([A,b])$, now let $Ax_0=b$.

then $[A, b] = [A, Ax_0] = A[I, x_0] \Rightarrow r([A,b]) \leq r(A)$

"$\Leftarrow$" let the $r = r(A) = r([A,b])$, and w.o.l.g. $a_1, \dots, a_r$ independent, then $a_1, \dots, a_r, b$ are dependent. We have nonzero

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \end{pmatrix} \neq 0, \text{ s.t. } [a_1, \dots, a_r, b] \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \end{bmatrix} = 0.$$

$x_{r+1} \neq 0$, otherwise $\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \neq 0$, but $[a_1, \cdots a_r] \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = 0$.

Then $b = A \begin{bmatrix} -x_1/x_{r+1} \\ \vdots \\ -x_r/x_{r+1} \end{bmatrix}$

16). $r(A^TA) = r(A)$. $r(A^TA) \leq r(A)$.

let $r = r(A)$, we show A^TA has r independent col's, w.o.l.g. $a_1, \dots, a_r$ independent.

We show $A^Ta_1, \dots, A^Ta_r$ are also independent.

In fact. $[A^Ta_1, \dots, A^Ta_r]x = 0 \Rightarrow$

$$\begin{pmatrix} a_1^T \\ \vdots \\ a_r^T \end{pmatrix} [a_1, \dots a_r] x = 0 \Rightarrow$$

$$x^T \begin{pmatrix} a_1^T \\ \vdots \\ a_r^T \end{pmatrix} [a_1, \dots, a_r] x = 0 \Rightarrow [a_1 \dots a_r] x = 0$$

$\Rightarrow x = 0$ since $a_1, \dots, a_r$ independent.

17). $A^TAx = A^Tb$ always has a solution.

$\forall b \quad r([A^TA, A^Tb]) = r(A^TA) = r(A)$

18). $r([A,B]) \leq r(A)+r(B)$.

Proof. Given any subset of columns of $[A,B]$ with size more that $(> r(A)+r(B))$ then either more than $r(A)$ col's come from A or more that $r(B)$ col's come from B. But ~~either~~ both of these ~~two sets are dependent~~, ~~hence this given subset of~~ $[A,B]$ is dependent.

19) $r\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = r(A)+r(B)$

$r\left(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right) \leq r(A)+r(B)+r(C)$, equality can be achieve.

20) If B, C are nonsingular matrices, then

$$r(BAC) = r(A)$$

This is a useful result, it implies that we can use elimination matrices B and C to simply A in order to find the rank of A.

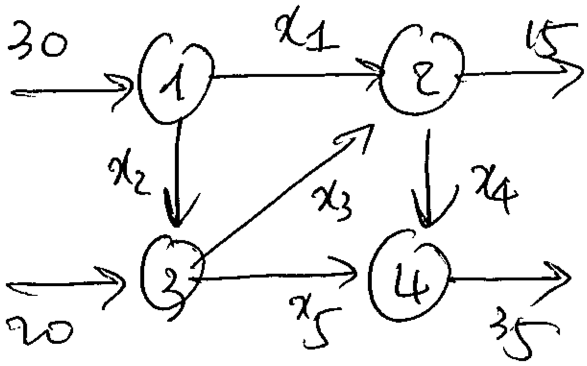
$$\mathbb{R}^{n\times n} \ni A = \left(\begin{array}{c|c} a_{11} & u^T \\ \hline a_1 & A_1 \end{array}\right) \longrightarrow \left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & A_1 - a_1 u^T/a_{11} \end{array}\right)$$

assume $a_{11} \neq 0$.

Then $r(A) = 1 + r\left(A_1 - a_1 u^T/a_{11}\right)$

In particular, if $r(A) = n$, then $A_1 - a_1 u^T/a_{11}$ is also nonsingular.

Flow of traffic through a road network



We have measurements on certain road segments and we want to figure out those on other segs.

Node 1: $x_1 + x_2 = 30$

Node 2: $x_1 + x_3 - x_4 = 15$

Node 3: $x_3 + x_5 - x_2 = 20$

Node 4: $x_4 + x_5 = 35$

In matrix format.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ 20 \\ 35 \end{bmatrix}$$

REF.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & -1 & 1 & 0 & 15 \\ 0 & 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & \cancel{-1} & \cancel{3}\,1 & 35 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 35 - x_3 - x_5 \\ x_2 = -20 + x_3 + x_5 \\ x_4 = 35 - x_5 \end{cases} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 35 \\ -20 \\ 0 \\ 35 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}$$