Lecture 14: Basis and Dimension

Span

Given a set of vectors: $v_1, \cdots v_n$. The span of $v_1, \cdots, v_n$: $\text{Span}\{v_1, \cdots, v_n\} = \{$all LCs of $v_1, \cdots, v_n\} = \{[v_1, \cdots, v_n]x : x \in \mathbb{R}^n\}$

For example $\text{Col}(A) = \text{Span}\{a_1, \cdots, a_n\}$, $A = [a_1, \cdots, a_n]$

Example $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{pmatrix} \in \mathbb{R}^{3\times 2}$, $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{pmatrix} \in \mathbb{R}^{2\times 3}$

$\text{Col}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} \right\} \Rightarrow$ independent col's

The row space of A, i.e., $\text{Col}(A^T) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$

dependent col's.

Spanning Set For a linear subspace $S \subset \mathbb{R}^m$, if

$S = \text{Span}\{v_1, \cdots, v_n\}$, then $v_1, \cdots v_n$ is called a *spanning set* of S. (Here, $v_1, \cdots, v_n$ are NOT required to independent). We say $v_1, \cdots v_n$ *span* the subspace S.

Example. When we compute the nullspace of A.

We write $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$ general sol of $Ax = 0$.

Then $N(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ ~~are~~ are the Spanning set of $N(A)$.

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but $\begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 2 \\ 0 \end{pmatrix}$ are also a Spanning set of $N(A)$

Two vectors can't span $\mathbb{R}^3$, but four vectors in $\mathbb{R}^3$ can't be independent. Need enough independent vectors to span the space and no more.

A <u>Basis</u> for a vector space is a collection of vectors
1) they are independent 2) they span the space

2) implies each vector in the space is a LC of its basis vectors, and there's only way to write this LC

Say $v_1, \dots, v_n$ are the basis, and $v = a_1 v_1 + \dots + a_n v_n$
$$= b_1 v_1 + \dots + b_n v_n$$
$$\Rightarrow (a_1 - b_1) v_1 + \dots + (a_n - b_n) v_n = 0$$
$\Rightarrow a_i - b_i = 0$ since $v_1, \dots, v_n$ are independent.

Example. the standard basis of $\mathbb{R}^3$ is
$$e_1, e_2, e_3, \quad \text{where } I_3 = [e_1, e_2, e_3]$$
Also the col's of every $n \times n$ invertible matrix is a basis of $\mathbb{R}^n$, because $\forall b$, $Ax = b$ has a unique sol.

$\mathbb{R}^n$ has ∞ # of bases!

Every set of independent vectors can be extended to a basis.

$v_1, \cdots v_n \in \mathbb{R}^m$, $n < m$, independent.

Use Gaussian elimination with pivoting

$$P[v_1, \cdots, v_n] = \left(\begin{array}{c|c} L_1 & \\ \hline L_2 & I_{m-n} \end{array}\right)\left(\begin{array}{c} U_1 \\ \hline 0 \end{array}\right)$$

Consider the matrix

$$P^T \left(\begin{array}{c|c} L_1 & 0 \\ \hline L_2 & I_{m-n} \end{array}\right)\left(\begin{array}{c|c} U_1 & 0 \\ \hline 0 & I_{m-n} \end{array}\right) \in \mathbb{R}^{m\times m}, \text{ nonsingular}$$

and has its first n col's equal to $v_1, \cdots, v_n$.

Every spanning set can be reduced to a basis.

In fact, take the col's of the spanning set to form a matrix, compute its REF, and the pivot col's indicate a ~~set~~ independent that form the basis.

Dimension of a space = the # of basis vectors in a basis of the space.

But we need to show two bases have the same # of basis vectors.

Th. If $v_1, \dots, v_n$, and $w_1, \dots, w_m$ are both bases of the same vector space, then $m=n$.

Proof. Suppose $n>m$. Since $\{v_1, \dots, v_n\}$ is a basis

$$W=[w_1, \dots, w_n] = [v_1, \dots, v_m]\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix} = VA$$

$A \in \mathbb{R}^{m\times n}$, and $n>m$, there's nonzero x, s.t. $Ax=0$

Then $Wx=0$, and $x\neq 0$. Contracting to col's of W independent.

Example. Basis for $N(A)$ from RREF of A.

Intersection of two subspaces.

S, T are two subspaces, then $S \cap T$ is also a subspace, E.g. S is a line going through the origin in $\mathbb{R}^3$, and T a plane going through the origin in $\mathbb{R}^3$. Then $S \cap T$ can be

1) $\{0\}$, 2) S, if $S \subset T$.

Sum of two subspaces

$$S+T = \{x+y : x \in S, y \in T\}$$

e.g. $S = C(A)$, $T = C(B)$. $S+T = C([A, B])$

Direct sum: if $S \cap T = \{0\}$ $S+T = S \oplus T$.

$$\dim(S) + \dim(T) = \dim(S+T) + \dim(S \cap T)$$

So $\dim(S) + \dim(T) = \dim(S \oplus T)$.

A set of vectors $v_1, \dots, v_s$ independent if

$$\alpha_1 v_1 + \dots + \alpha_s v_s = 0 \Rightarrow \alpha_1 = \dots = \alpha_s = 0.$$

$\dim(S)$ = max # of independent vectors of S.

E.g. the set of 2x2 matrices $:= M_2$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

are independent. $\Rightarrow \dim(M_2) = 4$.

Change of basis

with $\dim(S) = s$

~~Given~~ a subspace $S \subset \mathbb{R}^m$, assume the col's of V_1 and V_2 form two bases of S, ~~and dim~~ then there's a unique nonsingular matrix $B \in \mathbb{R}^{s\times s}$, s.t.

$$V_1 = V_2 B$$

Proof. Suppose $V_1 = V_2 B_1$, $V_1 = V_2 B_2$

$0 = V_1(B_1 - B_2)$, because $N(V_1) = \{0\} \Rightarrow B_1 = B_2$

The existence of B is a result of the definition of a basis. Independence of basis implies nonsingularity of B.

Example When we write $N(A) = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\8\end{pmatrix}\right\}$

The col's of the matrix $\underbrace{\begin{pmatrix}1 & 4\\2 & 5\\3 & 8\end{pmatrix}B}$, where B is an arbitrary nonsingular matrix

is also a basis of $N(A)$

Review

For $Ax=b$, $A\in\mathbb{R}^{m\times n}$, if the rows of A are dependent, then some of the equations in $Ax=b$ are actually redundant and can be removed without affecting the sol's of $Ax=b$.

The maximal # of rows to retain is $r=r(A)$ and RREF of A makes this choice explicit.

$$
\begin{array}{c}
r\left\{\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right. \\
m-r
\end{array}
\begin{pmatrix}
0\cdots0\,1\,*\cdots*\,0\,*\cdots*\,0\,*\cdots*\cdots*\,0\,*\cdots*\\
\qquad\quad 1\,*\cdots*\,0\,*\cdots*\cdots0\,*\cdots*\\
\qquad\qquad\qquad 1\,*\cdots*\cdots\vdots\quad\vdots\\
\qquad\qquad\qquad\qquad\ddots\quad\vdots\quad\vdots\\
\qquad\qquad\qquad\qquad\qquad 1\,*\cdots\\
\hline
0\quad\cdots\qquad\qquad\qquad 0
\end{pmatrix}
=R_0=\begin{pmatrix}R\\0\end{pmatrix}
$$

The # of removed equations are ~~those indicated~~ exactly $m-r$. indicated by the $m-r$ zero rows of RREF.

Example: Augmented matrix

$$[A, b] = \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{array}\right) \to$$

$$\to \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ \hline 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right) \to (R_0, \hat{b})$$

$Ax = b$ has a solution iff $b_3 - b_1 - b_2 = 0$.

The current syst. of equasions based RREF can be written as

$$\begin{cases} x_1 + 3x_2 + 2x_4 = b_1 \\ x_3 + 4x_4 = b_2 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = b_1 - 3x_2 - 2x_4 \\ x_3 = b_2 - x_3 \end{cases}$$

pivot variables: x_1, x_3

free variables: x_2, x_4.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \end{array}\right)$$

$$\underline{\underline{x_1}} + 3x_2 + 0\cdot x_3 + 2x_4 = b_1$$

$$0\cdot x_1 + 0\cdot x_2 + \underline{\underline{x_3}} + 4x_4 = b_2$$

$$x_1 = b_1 - 3x_2 - 2x_4.$$

$$x_3 = b_2 - 4x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 - 3x_2 - 2x_4 \\ x_2 \\ b_2 - 4x_4 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Example $A=\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{pmatrix}$, $b=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & -1 & b_3+2b_1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2b_1-b_2 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_3+b_1+b_2 \end{array}\right)$$

$Ax=b$ has a sol iff. $b_3+b_1+b_2=0$.

A has full col rank, $r=n=2$. If $b_1+b_2+b_3=0$.

Sol is unique $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2b_1-b_2 \\ b_2-b_1 \end{pmatrix}$

RREF: $R_0=\begin{pmatrix} I \\ 0 \end{pmatrix}$, $R=I$

Sumary R_0: $[I]$ $[I, F]$ $\begin{bmatrix} I \\ 0 \end{bmatrix}$ $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$A \in R^{m\times n}$ $r=m=n$ $r=m<n$ $r=n<m$ $r<m, r<n$

# of sol of $Ax=b$ 1 ∞ 0 or 1 0 or ∞

$(\hat{c}=0 \Rightarrow 1)$

$$\begin{pmatrix} I_r & F \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_p \\ x_F \end{pmatrix} = \begin{pmatrix} \hat{b} \\ \hat{c} \end{pmatrix}$$