lecture 15, RREF and four subspaces.

Given $A \in \mathbb{R}^{m\times n}$, the four subspaces are

1. Col space of A: $C(A) \subset \mathbb{R}^m$, $\dim(C(A)) = r$.
2. Null space of A: $N(A) \subset \mathbb{R}^n$, $\dim(N(A)) = n-$
3. row space of A: $C(A^T) \subset \mathbb{R}^n$, $\dim(C(A^T)) = r$
4. left nullspace of A: $N(A^T) \subset \mathbb{R}^m$, $\dim(N(A^T)) = m-$

where $r = \operatorname{rank}(A) \leq \min\{m, n\}$

Example

RREF form $R_0 = \begin{pmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad m=3,\ n=7$

$m=3,\ n=5,\ r=2.$

$C(A)$:

1. pivot cols are col 1 and col 4.

If $A = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^{3\times 5}$, then

$$C(A) = \text{Span}\{a_1, a_4\}$$

2. $N(A)$: consider $R_0 x = 0$.

$$x_1 + 3x_2 + 5x_3 + 7x_5 = 0.$$

$$x_4 + 2x_5 = 0.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3 & -5 & -7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix}$$

$$N(A) = \text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$\dim(N(A)) = n - r = 5 - 2 = 3.$$

$$R_0^T = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 0 \end{pmatrix}$$

3. $\dim(N(R_0^T)) = \dim(AT)$

$$R_0^T \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0. \quad \begin{cases} y_1 = 0. \\ 3y_1 = 0 \\ 5y_1 = 0 \\ y_2 = 0 \\ 7y_1 + 2y_2 = 0. \end{cases} \Rightarrow \begin{array}{l} y_1 = 0 \\ y_2 = 0. \end{array}$$

$$N(R_0^T) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim(AT) = \dim(N(R_0^T)) = 1.$$

4. $WA = R_0$. W nonsingular

$$A^T = R_0^T W^{-T}, \quad C(A^T) = C(R_0^T)$$

$$C(A^T) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$\dim(C(A^T)) = 2 = r.$$

Fundamental Th. of LA.

$A \in R^{m \times n}$, then

$$C(A) \oplus N(A^T) = R^m$$

$$C(A^T) \oplus N(A) = R^n$$

This is to say: 1) $C(A) \perp N(A^T)$

2) $C(A^T) \perp N(A)$.

<u>Proof</u>. let $a \in C(A)$, then $a = Ax_0$,

for any $b \in N(A^T)$, $b^T a = b^T A x_0$

$$= (\underbrace{A^T b}_{=0})^T x_0 = 0.$$

2) can be proved similarly

For two subspaces S, and T: $S \perp T$ means

$\forall x \in S, y \in T, \quad x^T y = 0$

Example. If $r(AB) = r(B)$, then for a compatible C

$$r(ABC) = r(BC)$$

this is because $N(B) \subset N(AB)$

Proof. $r(AB) = r(B)$, then ~~not~~ $N(AB) = N(B)$

Now if $ABCx = 0. \Rightarrow Cx \in N(AB) \Rightarrow Cx \in N(B)$

$BCx = 0. \quad x \in N(BC)$

Ex. A, B square matrices $AB = BA = 0$. $r(A^2) = r(A)$

then $r(A+B) = r(A) + r(B)$

A more detailed Derivation of ~~Lecture 13~~ RREF and Solving $Ax=b$.

Two basic tools

1) block elimination matrices: 2×2 block matrices

$$\left(\begin{array}{c|c} 1 & \\ \hline -l_1 & I \end{array}\right)\left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline a_1 & A_1 \end{array}\right) \equiv E_1 A.$$

when $a_{11} \neq 0$, set $l_1 = a_1/a_{11}$, the above product gives

$$\left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & A_1 - l_1 u_1^T \end{array}\right)$$

Similarly

$$\left(\begin{array}{c|c} I & -l_1 \\ \hline & 1 \end{array}\right)\left(\begin{array}{c|c} A_1 & u_1 \\ \hline a_1^T & a_{nn} \end{array}\right)$$

when $a_{nn} \neq 0$, set $l_1 = u_1/a_{nn}$, the above product becomes

$$\left(\begin{array}{c|c} A_1 - l_1 a_1^T & 0 \\ \hline a_1^T & a_{nn} \end{array}\right)$$

2) transposition matrices

$P_1 = (1, n_1)$ indicating the swapping of 1 and n_1. When apply P_1 to the left of A, $P_1 A$ is the result of swapping row 1 and row n_1 of A.

We will use the above two tools to construct REF and RREF of $A \in \mathbb{R}^{m \times n}$

One key difference from GE for nonsingular A is we need to find the next nonzero column in the current submatrix, rather than just use its 1st col.

Suppose we have done $k-1$ steps of elimination, the reduced matrix has the form

$$E_{k-1}P_{k-1}\cdots E_1P_1A = \left(\begin{array}{c|c} * & * \\ \hline 0 & \boxed{\begin{array}{c|c|c} 0 & \;\; & \;\; \end{array}} \end{array}\right)$$

current submatrix.

n_k

where we indicate n_k (overall col number) as the 1st nonzero col in the current submatrix.

We can use transposition and block elimination to reduce the current submatrix to

$$\left(\begin{array}{c|c|c} 0 & \alpha_k & * \cdots * \\ \cline{2-3} & 0 & * \end{array}\right)$$

let P_k, E_k be the corresponding transposition matrix, and block elimination matrix.

$$E_k P_k \left(E_{k-1} P_{k-1} \cdots E_1 P_1 A \right) = \left[\begin{array}{c|c|c|c|c} & \alpha_1 & * & \cdots & * \\ \hline 0 & 0 & \alpha_2 & * \cdots & * \\ \hline & & 0 & \alpha_k & * \cdots * \\ \hline & & & 0 & \end{array}\right]$$

This process will terminate when the current submatrix is all zero (or α_k is either in the last col or last row)

Let us assume the process terminates after r steps.

$$E_r P_r \cdots E_1 P_1 A$$

$$= \begin{array}{c} n_1 \cdots n_2 \cdots n_r \\ \left[\begin{array}{c|c|c} 0 & \alpha_1 \; * \cdots & * \\ \hline 0 & 0 & \alpha_2 \; * \cdots * \\ & & \ddots \quad \alpha_r \; * \cdots * \\ \hline 0 & 0 & 0 \end{array}\right] \end{array} \equiv U_0.$$

is what we call row echelon form REF.

U_0 has exactly r nonzero rows,

nonzero pivot α_k starts row k, and its col # is n_k.

all the zero rows are below the nonzero rows.

$n_1, n_2, \dots, n_r$ indicate the first r independent col's of A, i.e. partition $A = (a_1, a_2, \dots, a_n)$

then $(a_{n_1}, a_{n_2}, \dots, a_{n_r})$ are independent.

and they are called pivot col's because each contains a nonzero pivot α_k.

with U_0, we can already solve $Ax=0$, i.e., finding $N(A)$, and solve $Ax=b$ if $b \in C(A)$. More examples to come

In $\mathcal{U}_0$, the pivot col's are scatter around, we can use column transpositions to move them to the first r col's of the metrix.

For example, $Q_1 = (1, n_1)$, when applied to the right of $\mathcal{U}_0$, exchange its col 1 and n_1

Then $\mathcal{U}_0 Q_1 \cdots Q_r$ with $Q_1 = (1, n_1), \cdots Q_r = (r, n_r)$

$$= \left(\begin{array}{c|c} \begin{matrix} \alpha_1 & & * \\ & \alpha_2 & \\ 0 & & \ddots \ \alpha_r \end{matrix} & * \\ \hline 0 & 0 \end{array}\right) \equiv \left(\begin{array}{c|c} \mathcal{U}_1 & \mathcal{U}_2 \\ \hline 0 & 0 \end{array}\right)$$

where $\mathcal{U}_1 \in R^{r \times r}$ is uppertri and nonsingular

Hence $E_r P_r \cdots E_1 P_1 A Q_1, \cdots Q_r = \left(\begin{array}{c|c} \mathcal{U}_1 & \mathcal{U}_2 \\ \hline 0 & 0 \end{array}\right)$

$C(A) = \operatorname{Span}\{C\}$ with dimension r

when we bring the cols of $\left(\begin{array}{c|c} I_r & F \\ \hline 0 & 0 \end{array}\right)$ to its original positions by post-multiply Q^T. the result is the reduced row echelon form RREF.

$$R_0 = \left(\begin{array}{c|c} I_r & F \\ \hline 0 & 0 \end{array}\right) Q^T = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$A = CR$ the so-called col-row decomposition.

R_0 has the form

$$\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & \cdots & n_r \end{array} \\ \left[\begin{array}{c|cccccc} & 1 & * & 0 \cdot & 0 * & & * \\ \hline & & & 1 & * & 0 & * \\ \hline & & & & & 0 & \\ & & & & & \vdots & \\ & & & & & 0 & \\ \hline & & & & & 1 & * \cdots * \\ \hline 0 & 0 & & 0 & & 0 & \end{array}\right] \end{array}, \quad I_n = (e_1, \cdots, e_n)$$

$\rightarrow$ pivot col n_k is simply e_k, $k = 1, \cdots, n$.