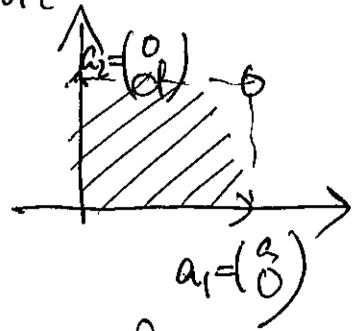
Lecture 19 Determinants

$A \in \mathbb{R}^{2\times 2}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad - bc$

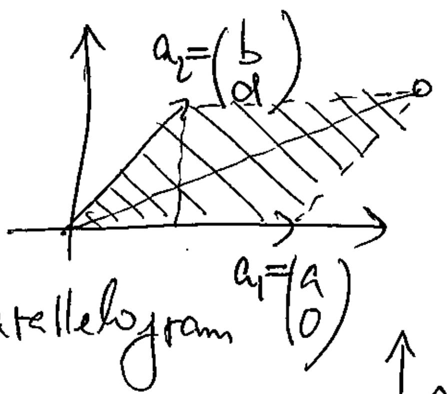
What's the geometric picture?

$A = \left(\begin{array}{c|c} a & 0 \\ 0 & d \end{array}\right)$
$\quad a_1 \quad a_2$

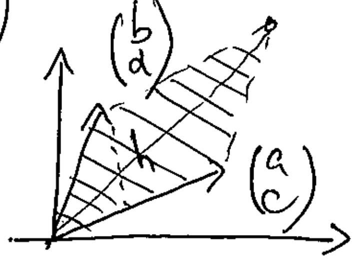


Area of rectangle $= ad$.

$A = \left(\begin{array}{c|c} a & b \\ 0 & d \end{array}\right)$
$\quad a_1 \quad a_2$



ad = Area of ~~para~~ parallelogram



Now for general $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

We can rotate $a_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ to $\begin{pmatrix} \|a_1\| \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} b \\ d \end{pmatrix}$ will be rotate to $\begin{pmatrix} * \\ h \end{pmatrix}$

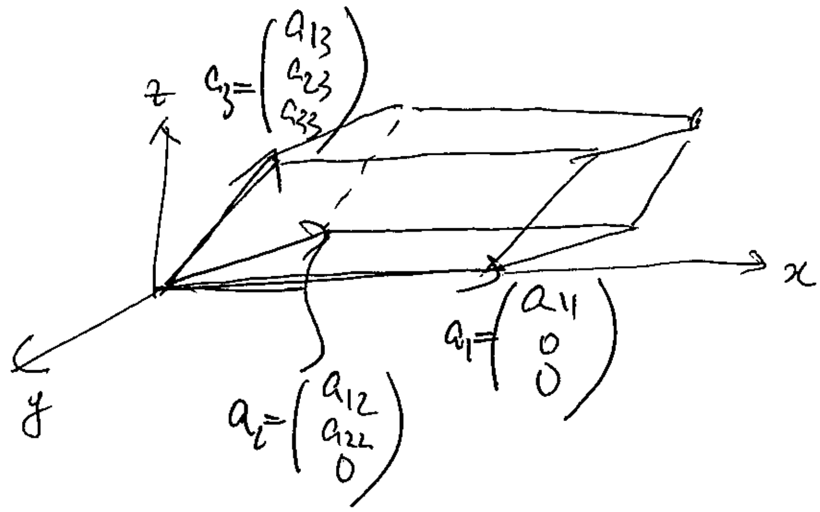
$\underbrace{\|a_1\| \cdot h}$ = Area of parallelogram, because rotation preserves area

It's not hard to extend the above argument to R^3. For example

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33}$$

The box with corners $a_1 = \begin{pmatrix} a_{11} \\ 0 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$
(or edges)

has volume $a_{11}a_{22}a_{33}$



Now for any $A = (a_1, a_2, a_3) \in R^{3\times 3}$, there's ortho.

Q: $QA = (Qa_1, Qa_2, Qa_3) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{pmatrix}$

Volume of box with corners $a_1, a_2, a_3 = |r_{11} r_{22} r_{33}|$

$= |\det A|$ absolute value of $\det A$

In general,

For $A = [a_1, \cdots, a_n] \in \mathbb{R}^{n \times n}$, $|\det A|$ is the volume of the box with corners $a_1, a_2, \cdots, a_n$.

We can also appeal to QR decomposition of A.

$$A = QR, \quad Q \text{ ortho. } R = \begin{pmatrix} r_{11} & & * \\ & \ddots & \\ 0 & & r_{nn} \end{pmatrix}.$$

$$|\det A| = |\underbrace{r_{11} \cdots r_{nn}}_{\text{product of diagonals of } R}|.$$

Properties of determinants

1. $\underline{\det}$: $\cancel{A^{n\times n}}$ $\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$. $\det A \in \mathbb{R}$.

   a mapping from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$, i.e. for any square matrix A, assign a real number $\det A$

$\Rightarrow$ 2. $\det(AB) = \det A \det B$

3. let $A = (a_1, \cdots, a_n)$, for scalar α.

$$\det(\alpha a_1, a_2, \cdots, a_n) = \alpha \det A.$$

Since $\det\begin{pmatrix} \alpha & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \alpha$, and $[\alpha a_1, \cdots, a_n] = A\begin{pmatrix} \alpha & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

4. pre- and post-multiply with elimination matrices $\underbrace{\text{do not}}$ change the determinant.

5. If A is nonsingular $\Rightarrow \det A^{-1} = (\det A)^{-1}$

6. $\det A^T = \det A$.

$\underline{\text{Proof}}$. $A = PLU$. $\det A = \det P \det L \det U$

$$= \det P \,(u_{11} \cdots u_{nn})$$

$A^T = U^T L^T P^T$, $\det A^T = \det P^T (u_{11} \cdots u_{nn})$

but $P^T = P^{-1}$ $\det P^T = (\det P)^{-1} = \det P$

since $\det P = \pm 1$.

$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$. Any permutasion = product of transpositions

(not necessarily unique, but ~~both~~ have the same ~~parity~~ parity)

7. A is nonsingular iff $\det A \neq 0$.

8. $\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \det B$ where A, B square

9. $\det [u+v, a_2, \cdots, a_n] = \det [u, a_2, \cdots, a_n]$
$+ \det [v, a_2, \cdots, a_n]$

Proof. If $\underbrace{a_2, \cdots, a_n}$ are dependent, then all 3 matrices are singular. $\det = 0$

So assume $a_2, \cdots, a_n$ independent, then if $u, a_2, \cdots, a_n$ dependent, and $v, a_2, \cdots, a_n$ dependent.

$\Rightarrow$ $u+v, a_2, \cdots, a_n$ dependent all 3 matrices have $\det = 0$

Now assume $u, a_2, \cdots, a_n$ $\underbrace{\text{independent}}$, then

$v = \alpha_1 u + \alpha_2 a_2 + \cdots + \alpha_n a_n$, a linear combination of the basis vectors

$$\det[u+v, a_2, \cdots, a_n]$$

use elimination!

$$= \det[(1+\alpha_1)u + \alpha_2 a_2 + \cdots + \alpha_n a_n, a_2, \cdots, a_n]$$

$$= \det[(1+\alpha_1)u, a_2, \cdots, a_n] = (1+\alpha_1)\det[u, a_2, \cdots, a_n]$$

$$\det[v, a_2, \cdots, a_n] = \det[\alpha_1 u + \alpha_2 a_2 + \cdots + \alpha_n a_n, a_2, \cdots, a_n]$$

$$= \det[\alpha_1 u, a_2, \cdots, a_n] = \alpha_1 \det[u, \alpha_2, \cdots, \alpha_n]$$

$$\det[u, a_2, \cdots, a_n] + \det[v, a_2, \cdots, a_n] = \underbrace{(1+\alpha_1)}\det[u, a_2, \cdots, a_n]$$

10. $$\det[\alpha u + \beta v, a_2, \cdots, a_n] = \alpha \det[u, a_2, \cdots, a_n] + \beta \det[v, a_2, \cdots, a_n]$$

Therefore $\det A$ is a $\underbrace{\text{multi-linear}}$ function considered as functions of $a_1, \cdots a_n$ (linear in each argument $a_1, \cdots, a_n$)

11. Exchanging two rows (or two cols) of a matrix changes the sign of its determinant.

12 <u>Cofactors</u>

Consider $A = (a_1, a_2, \cdots, a_n)$.

assume: $a_1 = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_{21} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n1} \end{pmatrix}$.

$$\det A = \det\left[\begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, a_2, \cdots, a_n\right] + \det\left[\begin{pmatrix} 0 \\ a_{21} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, a_2, \cdots, a_n\right]$$

$$+ \cdots + \det\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n1} \end{pmatrix}, a_2, \cdots, a_n\right], \quad \text{expansion}$$

Now $\det\left[\begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, a_2, \cdots, a_n\right] = a_{11} \det A(1,1)$

where <u>$A(1,1)$</u> is A with its 1st row and 1 col removed

$\det\left[\begin{pmatrix} 0 \\ a_{21} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, a_2, \cdots, a_n\right] = (-1)\, a_{21} \det A(2,1)$

<u>$A(2,1)$</u> is A with its 2nd row and 1 col removed.

, and $\det\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n1} \end{pmatrix}, a_2, \cdots, a_n\right] = (-1)^{n-1} a_{n1} \det A(n,1)$.

Definition $\qquad$ 代数余子式

Cofactor $C_{ij} = (-1)^{i+j} \underbrace{\det A(i,j)}$ 余子式

where $A(i,j)$ is A with its i-th row, and j-th col removed.

$\det A = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}$, expansion

more generally, $\det A = a_{1k}C_{1k} + \cdots + a_{nk}C_{nk}$

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let $C = (C_{ij}) \in R^{n\times n}$, then.

$(\det A) I = AC^T \Rightarrow A^{-1} = \underbrace{\frac{1}{\det A}} C^T$

Proof. $\quad a_{1j}C_{1k} + \cdots + a_{nj}C_{nk} = 0 \quad j \neq k$.

because $a_{1j}C_{1k} + \cdots + a_{nj}C_{nk} = \det[a_1, \cdots, a_j, \cdots a_n]$

$\uparrow$ k-th col.

$\Rightarrow$ Big formula for $\det A$ for each permutation matrix P.

it has n "1"s, one in each col / row. form the product

$\pm a_{1s_1}a_{2s_2}\cdots a_{ns_n} \Rightarrow \det A = \sum \pm a_{1s_1}\cdots a_{ns_n}$

where $\begin{pmatrix} 1 & 2 & \cdots & n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix}$ a permutation. $\pm$ is determined by the parity

Cramer's rule

let $A(u,i)$ be matrix A with its i-th col replaced by the vector u, then

$$\det A(u,i) = u_1 C_{1i} + \cdots + u_n C_{ni}, \text{ expansion}$$

C_{ij} the (i,j)-cofactor of A, and $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

Now from $Ax = b$, and $x = A^{-1}b$

$$= \frac{1}{\det A} C^T b = \begin{pmatrix} b_1 C_{11} + \cdots + b_n C_{n1} \\ \vdots \\ b_1 C_{1n} + \cdots + b_n C_{nn} \end{pmatrix} \frac{1}{\det A}$$

$$= \frac{1}{\det A} \begin{pmatrix} \det A(b,1) \\ \vdots \\ \det A(b,n) \end{pmatrix}$$

Therefore, if A nonsingular, $Ax = b$ has sol $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$x_i = \frac{\det A(b,i)}{\det A}, \quad i = 1, \cdots, n.$$

Prob 1. $\det(I+uv^T) = 1+u^Tv$

Proof. Use householder, $Hu = \|u\| e_1$.

$$\det(I+uv^T) = \det(H(I+uv^T)H)$$

$= \det(I + \|u\| e_1 (Hv)^T)$ the matrix is upper tri.

$= 1 + \|u\| e_1^T Hv$, but $He_1 = u/\|u\|$

$= 1 + u^Tv$

In particular, $\det(I - 2uu^T) = -1$, with $\|u\|=1$

Prob 2. Assume $a_i \neq 0$, $i = 1, \cdots, n$.

$$\det\begin{pmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{pmatrix} = \det\left(\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots & \\ & & & a_n \end{pmatrix} + \mathbb{1}\mathbb{1}^T\right)$$

$$= (a_1 \cdots a_n) \det\left(I + \begin{pmatrix} a_1^{-1} & & \\ & \ddots & \\ & & a_n^{-1} \end{pmatrix} \mathbb{1}\mathbb{1}^T\right)$$

$= (a_1 \cdots a_n)\left(1 + \sum_{i=1}^{n} \frac{1}{a_i}\right)$, what will be the result if some $a_i = 0$?

Prob 3. If A is nonsingular,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B).$$

Proof. block elimination $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$

Prob 4. $\det(I_n - AB) = \det(I_m - BA)$

where $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$

Prob 5. Hadamard's inequality

$$\det A \leq \|a_1\| \, \|a_2\| \cdots \|a_n\|.$$

Prob. 6. let C be the cofactor matrix, then

$$\det C = (\det A)^{n-1}.$$

Proof $A C^T = (\det A) I$.

Extra material!

Determinants via QR decomposition. (outline)

If A is singular, define $\det A = 0$, and we only consider nonsingular matrices.

1. QR decomposition. There are unique Q orth. and R upper tri with positive diagonal elements. s.t.

$$A = QR.$$

2. For an ortho Q, Q can be written as the product of a sequence of rotations and a permutation matrix. Define $\det Q$ as $+1$ or -1 according to the parity of the permutation matrix.

3. Define $\det A = (r_{11} \cdots r_{nn}) \cdot (\text{parity of } Q)$.

We now outline a proof $\det AB = \det A \det B$

1) If $A = QL$, where L has positive diagonals then $\det A = (l_{11} \cdots l_{nn})(\text{parity}(Q))$.

2) $\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \det B$

Proof. $A = Q_A R_A$, $\cancel{B = Q_B R_B}$, $C = Q_C R_C$

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} Q_A & \\ & Q_B \end{pmatrix} \begin{pmatrix} R_A & Q_A^T B \\ 0 & R_B \end{pmatrix}$$

$$\text{parity}\left(\begin{pmatrix} Q_A & \\ & Q_B \end{pmatrix} \right) = \text{parity}(Q_A)\, \text{parity}(Q_B).$$

3) pre-multiply with $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ change the sign of the determinant. As a consequence $\det(-A) = (-1)^n \det A$.

$$\det \left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] = (-1)^n \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

4) $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Proof. Use QR.

Now $\begin{pmatrix} I & B \\ A & 0 \end{pmatrix} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & -AB \end{pmatrix}$

implies $(-1)^n \det A \det B = (-1)^n \det AB$