Lecture 2

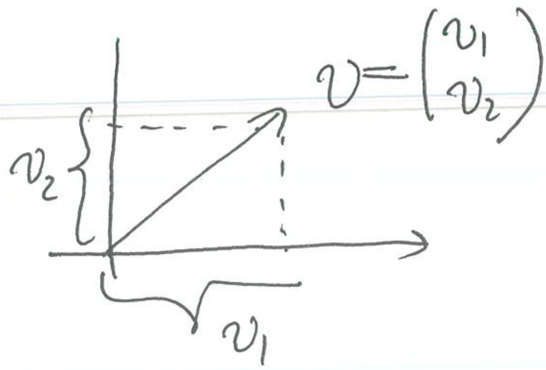
1. Dot-products

$u, v \in \mathbb{R}^n$, $\quad u \cdot v \equiv u_1 v_1 + \cdots + u_n v_n$.

The length of vector $v = (v_1^2 + \cdots + v_n^2)^{1/2}$

$= (v \cdot v)^{1/2} \equiv \|v\|$ (norm of v)

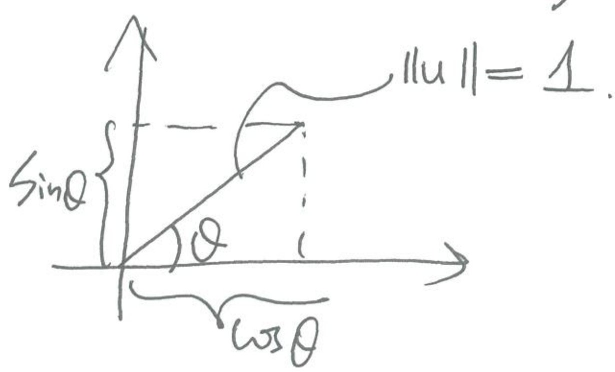
In 2d-space



When $\|v\|=1$, v is called a unit vector

for $v \neq 0$ (the zero vector), $\dfrac{v}{\|v\|}$ is a unit vector

In 2d-space $\quad u = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$



If we define

$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then $i \cdot u = \cos\theta$

$j \cdot u = \sin\theta$

2. A toy example: A trip to the supermarket.

	Apple	Orange	Banana	Total
Quantity in kg	3	2	4	
unit price	5	3.5	2	
cost	15	7	8	30

$$w = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} 5 \\ 3.5 \\ 2 \end{pmatrix} \quad \text{Total cost} = w \cdot v = 30.$$

weight vector   feature vector

Other related notations: let $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

1) $\mathbb{1} \cdot w = 9$ total weight.

2) $\left(\frac{\mathbb{1}}{n}\right) \cdot w$ average

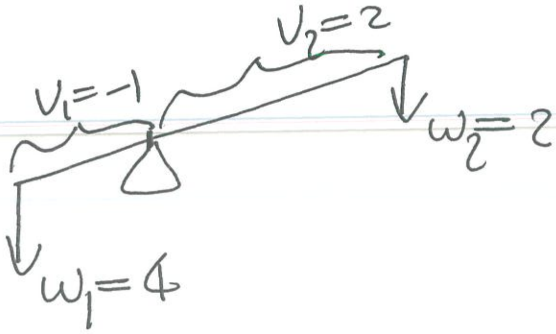
3) sum of squares $w \cdot w$

3. Perpendicular vectors

$v, w \in \mathbb{R}^n$, if $v \cdot w = 0$, v and w ~~are~~ are perpendicular, and we write $v \perp w$ or v is orthogonal to w.

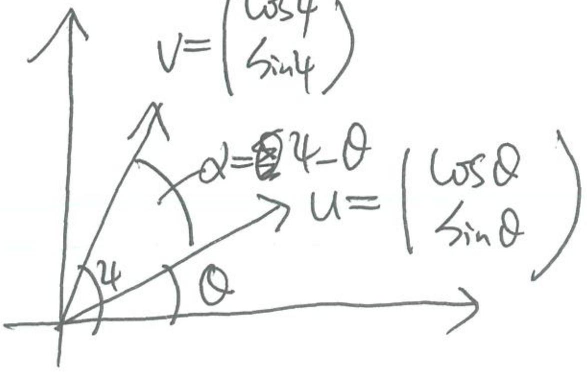
<u>Theorem.</u> If $v \perp w$, then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$

$w = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



$w \perp v$.

4. let two unit vectors $u = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $v = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$



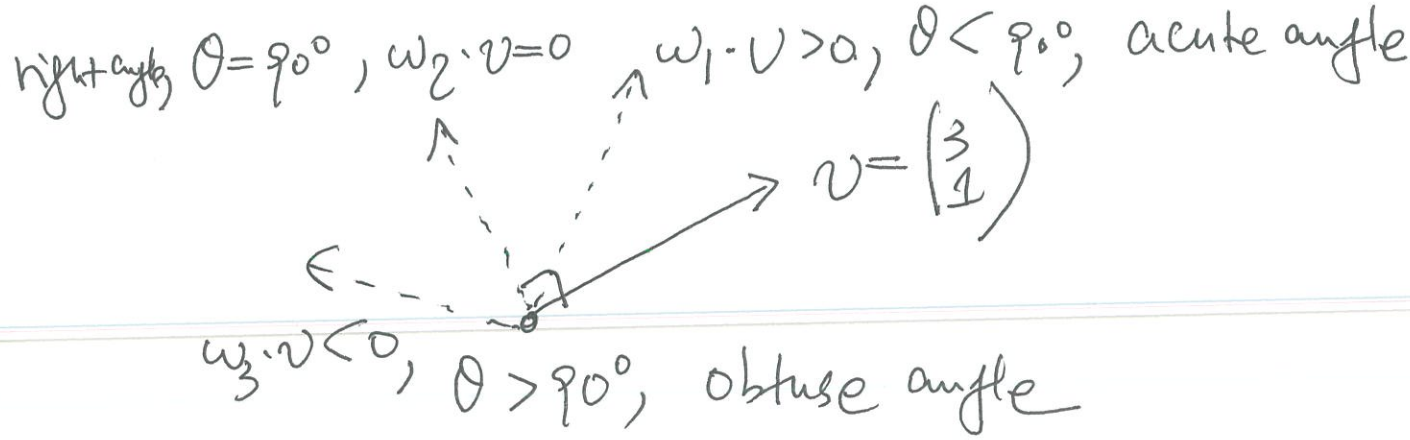
$u \cdot v =$

$\cos\theta\cos\psi + \sin\theta\sin\psi$

$= \cos(\underbrace{\psi - \theta}_{\alpha})$

The ~~sing~~ sign of $u \cdot v$:

$$u \cdot v = \begin{cases} > 0 & \text{angle } \alpha < 90^\circ \\ = 0 & \alpha = 90^\circ \\ > 0 & \alpha > 90^\circ \end{cases}$$



$\underline{\underline{5}}$. In n-dimensional space, u, v two unit vectors, define $\cos\theta = u \cdot v$

θ is the angle between u and v

Cosine formula: $\cos\theta = \left(\frac{u}{\|u\|}\right) \cdot \left(\frac{v}{\|v\|}\right)$

for two non-zero vectors u and v.

Example term-doc matrix.

	doc_1	doc_2	$\cdots$	doc_n
$word_1$	2	0		10
$\vdots$	$\vdots$		$\cdots$	
$word_m$	3	2		10

Similarity between two documents defined to be

$$\left(\frac{doc_1}{\|doc_1\|}\right)\cdot\left(\frac{doc_2}{\|doc_2\|}\right)$$

<u>6</u> Cauchy-Schwarz Inequality.

$$|u\cdot v| \leq \|u\|\,\|v\|$$

$$\Longleftrightarrow \quad (u_1v_1+\cdots+u_nv_n)^2 \leq (u_1^2+\cdots+u_n^2)(v_1^2+\cdots+v_n^2)$$

A tricky proof: $\quad 0 \leq \big\| \|u\|v - \|v\|u \big\|^2$

$$= 2\|u\|^2\|v\|^2 - 2\|u\|\|v\|\, u\cdot v$$

1) Cauchy–Schwarz Inequality (An usual proof)

$|u \cdot v| \leqslant \|u\| \, \|v\|$.

$(u_1 v_1 + \cdots + u_n v_n)^2 \leqslant (u_1^2 + \cdots + u_n^2)(v_1^2 + \cdots + v_n^2)$

Proof $\|u\| = 1, \ \|v\| = 1$. $\qquad |x+y| \leqslant |x| + |y|$.

$|u_1 v_1 + \cdots + u_n v_n| \leqslant |u_1 v_1| + \cdots + |u_n v_n|$.

$|u_i v_i| \leqslant \frac{1}{2}(u_i^2 + v_i^2)$ because

$\frac{1}{2}(|u_i| - |v_i|)^2 \geqslant 0$.

$\leqslant \frac{1}{2}(u_1^2 + v_1^2) + \cdots + \frac{1}{2}(u_n^2 + v_n^2)$

$= \frac{1}{2}(u_1^2 + \cdots + u_n^2) + \frac{1}{2}(v_1^2 + \cdots + v_n^2) = \frac{1}{2} + \frac{1}{2} = 1$.

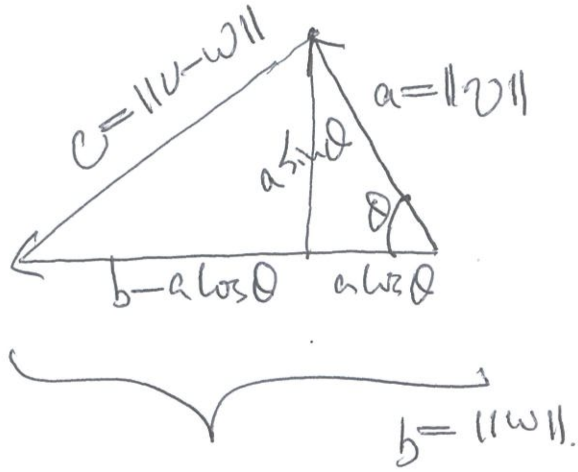
2) Triangle inequality: $\|v + w\| \leqslant \|v\| + \|w\|$.

$\|v + w\|^2 = \|v\|^2 + \|w\|^2 + 2\, v \cdot w$

$\leqslant \|v\|^2 + \|w\|^2 + 2\|v\| \, \|w\|$

$= (\|v\| + \|w\|)^2$

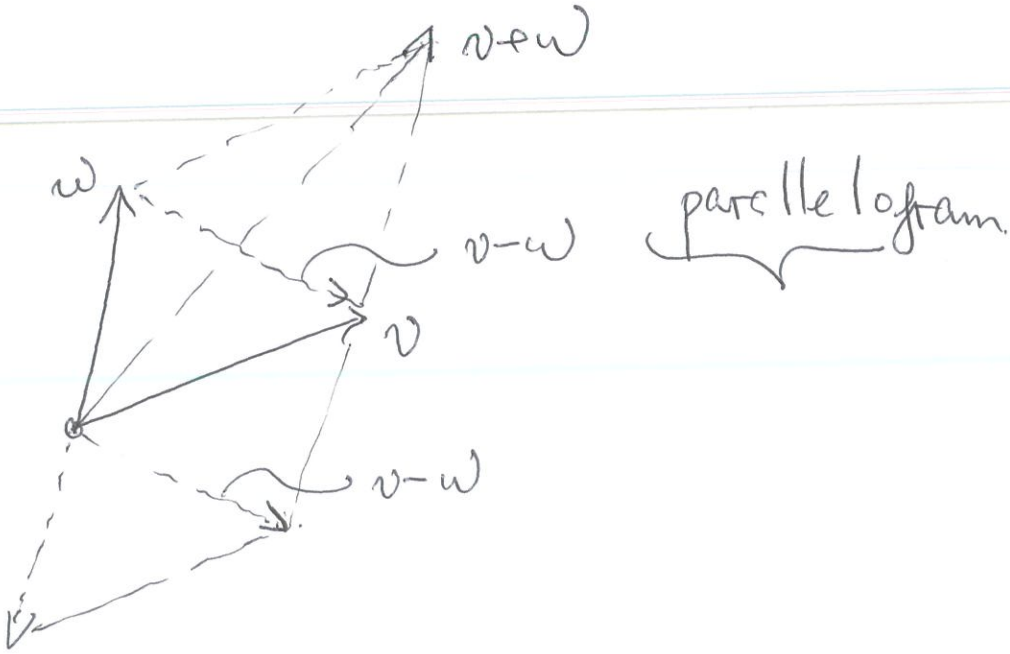
题 7



$$c^2 = (b - a\cos\theta)^2 + a\sin^2\theta = a^2 + b^2 - 2ab\cos\theta$$

$$\|v - w\|^2 = \|v\|^2 + \|w\|^2 - 2\, v\cdot w \Rightarrow v\cdot w = \|v\|\|w\|\cos\theta$$

题 8



$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

题 9 $Ax = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} x = \begin{bmatrix} A_1\cdot x \\ \vdots \\ A_m\cdot x \end{bmatrix}$

$$[a_1, \dots, a_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1 + \dots + x_na_n$$