Lecture 20 Linear Transformations (or Maps)

Review. A **map** (映射) between two sets S and T. has three parts:

1) S: domain 2) T: co-domain, 3)

f: correspondance that assigns $f(s) \in T$ for each $s \in S$.

(We use maps and transformations interchangably)

Definition

A linear map / transformation between two vector spaces is a map T between $\underbrace{V \text{ and } W}_{\text{general vector spaces}}$, s.t. (it's easy to check $\underline{T(0)=0}$)

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \ \forall u, v \in V.$$

Example. let $V = R^n$, $W = R^m$, $T(u) = Au$, where $A \in R^{m \times n}$

Then $T(\alpha u + \beta v) = A(\alpha u + \beta v) = \alpha Au + \beta Av = \alpha T(u) + \beta T(v)$

More concretely, $n=2$, $m=3$, and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 2x+5y \\ 7x+9y \end{pmatrix} \iff T(u) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = Au$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Many transformations are <u>NOT</u> linear.

<u>Example</u> $T(u) = u + u_0$, where $u_0 \neq 0$ is a fixed vector

$$T(\alpha u + \beta v) = (\alpha u + \beta v) + u_0, \text{ while}$$

$$\alpha T(u) = \alpha(u + u_0) \quad \beta T(v) = \beta(v + u_0)$$

$$\alpha T(u) + \beta T(v) = \alpha u + \beta v + \underbrace{\alpha u_0 + \beta u_0}_{\neq u_0, \text{ generally.}}$$

$T(u) = u + u_0$ is an <u>affine</u> map. $\left(\begin{array}{l} T(\alpha u + (1-\alpha) v) \\ = \alpha T(u) + (1-\alpha) T(v) \end{array} \right)$

<u>Example</u> $T(u) = \|u\|$ is not linear.

1) $T(u+v) = \|u+v\| \leqslant \|u\| + \|v\|$.

2) $T(-u) = \|-u\| = \|u\|$ (rather than $(-1)\|u\|$).

A linear map is determined by its values on ~~the~~ a set of basis vectors $\{u_1, \dots, u_n\}$ of V, since $\forall u$, $u = \sum \alpha_i u_i$

$$T(u) = T\left(\sum \alpha_i u_i\right) = \sum \alpha_i \underline{\underline{T(u_i)}}.$$

Geometry of linear maps.

T maps the line connecting two vectors u and v

to the line connecting T(u) and T(v) also.

$$\forall\, 0<\alpha<1, \quad T\big(\underbrace{\alpha u+(1-\alpha)v}_{\text{a point on the line segment}}\big)=\alpha T(u)+(1-\alpha)T(v).$$

T maps a triangle with three corners u, v, w to

a triangle with three corners T(u), T(v), T(w) also

for $w_1 \geqslant 0,\ w_2 \geqslant 0,\ w_3 \geqslant 0 \quad w_1+w_2+w_3=1.$

$$T\big(\underbrace{w_1 u+w_2 v+w_3 w}_{\text{a point on the original triangle}}\big)=\underbrace{w_1 T(u)+w_2 T(v)+w_3 T(w)}_{\text{a point on the transformed triangle}}.$$

Matrix representation of a linear map

1. Matrix representation is always w.r.t a pair of (with respect to) bases: $\{v_1, \dots, v_n\}$ for V, and $\{w_1, \dots, w_m\}$ for W.

Since $T(v_i) \in W$, $T(v_i) = \cancel{[w_1, \dots, w_m]} \sum a_{ji} w_j$

$\left(= \{w_1, \dots, w_m\} \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}, \right.$ ~~symbolically~~ Symbolically $\Big)$

Again, symbolically,

$$T\{v_1, \dots, v_n\} = \{w_1, \dots, w_m\} \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A.}$$

A is the matrix representation of T wrt. to the two bases: $\{v_1, \dots, v_n\}$, and $\{w_1, \dots, w_m\}$.

$A \in R^{m \times n}$, $\dim(V) = n$, $\dim(W) = m$.

Example. take the standard bases

$$\mathbb{R}^2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad \mathbb{R}^3 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 2x+5y \\ 7x+9y \end{pmatrix}$ satisfies

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 9 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The matrix representation for T w.r.t. the standard bases

is $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$, ~~and~~

Change of basis

New basis

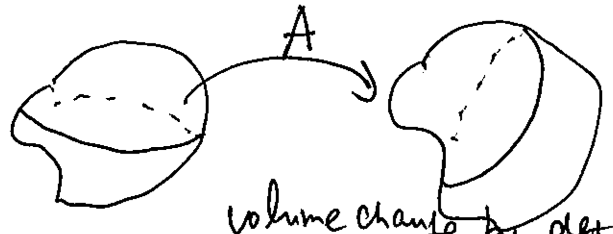
$$\{\tilde{v}_1, \dots \tilde{v}_n\} = \{v_1, \dots, v_n\} B_{in}$$

$$\{\tilde{w}_1, \dots \tilde{w}_m\} = \{w_1, \dots, w_m\} B_{out}.$$

($\overset{n\times n}{B_{in}}, \overset{m\times m}{B_{out}}$ are nonsingular)

The representation for T w.r.t. the new bases is

$$\left(\overset{m\times m}{B_{out}^{-1}} \overset{m\times n}{A} \overset{n\times n}{B_{in}} \right)$$

Symbolically, 

volume change by $\underline{\det A}$.

$$T\{v_1, \dots, v_n\} = \{w_1, \dots, w_m\} A$$

$$T\{v_1, \dots, v_n\} B_{in} = \{w_1, \dots, w_m\} B_{out} \left(B_{out}^{-1} A B_{in}\right)$$

$$T\{\tilde{v}_1, \dots, \tilde{v}_n\} = \{\tilde{w}_1 \cdots \tilde{w}_m\} \underbrace{B_{out}^{-1} A B_{in}}.$$

A large part of linear algebra is about
$\underline{\underline{\text{searching for good bases.}}}$

1) $V = W$, and A diagonalizable $X^{-1} A X = \Lambda$ diagon

2) $U^T A V = \Sigma$, SVD of A. (X the set of eigenvec of A

3) $V = W$, general A. $X^{-1} A X = J$ $\underbrace{\text{Jordan form}}$

<u>Change of Volume</u>

Consider $a_i \in \mathbb{R}^n$, $\det \underbrace{[a_1, \dots, a_n]}_{A}$ is the signed Volume of the box with corners at a_i's. Now for $B \in \mathbb{R}^{n \times n}$

$[Ba_1, \dots, Ba_n] = B[a_1, \dots, a_n]$ is the box with corners at Ba_i's
$= BA$

Since $\det(BA) = \det B \det A$. The volume of the new box is $\underline{\det B}$ times that of the original