lecture 21. Linear maps and Examples

Compositions of maps.

$$S: A \to B, \quad T: B \to C$$

$T \circ S(a) = T(S(a))$ composition of S and T.

$\Rightarrow$ $T \circ S: A \to C$

Now for two linear maps:

$S: \mathcal{U} \to V$, $T: V \to W$. $\mathcal{U}, V, W$ vector spaces

B ~~A~~ is the ~~map~~ matrix representation

$$S\{u_1, \dots, u_n\} = \{v_1, \dots v_m\} \cancel{A} B$$

A ~~B~~ is the matrix representation

$$T\{v_1, \dots v_m\} = \{w_1, \dots, w_p\} \cancel{B} A.$$

Then $T \circ S\{u_1 \dots u_n\} = T(\{v_1 \dots v_m\} \cancel{A} B)$

Symbolically $= \{w_1, \dots, w_p\} \underbrace{AB}$

More precisely, $T \circ S(u_i) = T(S(u_i))$ ~~(?)~~

$$S(u_i) = [v_1, \dots, v_m](Be_i)$$ linear comb. of the basis with coefficients Be_i

$$= B_{1i} v_1 + \dots + B_{mi} v_m$$

$$T(S(u_i)) = B_{1i} T(v_1) + \dots + B_{mi} T(v_m)$$

$$= \left[[w_1, \dots, w_p] A e_1, \dots, [w_1, \dots, w_p] A e_m \right] \begin{bmatrix} B_{1i} \\ \vdots \\ B_{mi} \end{bmatrix}$$

$$= \{w_1, \dots, w_p\} A B e_i$$

<u>Example</u>. $A \in \mathbb{R}^{3\times4}$, $r(A) = 2$. $T(v) = Av$.

Choose a basis $\{v_1, v_2, v_3, v_4\}$ for $\mathbb{R}^4$ as follows:

v_1, v_2 are from $C(A^T)$, and v_3, v_4 are from $N(A)$

Choose a basis $\{w_1, w_2, w_3\}$ for $\mathbb{R}^3$ as follows:

$w_1 = Av_1$, $w_2 = Av_2$, and $w_3 \in N(A^T)$.

What's the matrix representation of T w.r.t. those two bases

$$T\{v_1, v_2, v_3, v_4\} = \{Av_1, Av_2, 0, 0\}$$ It can be shown Av_1, Av_2 are independent

$$= \{\underset{=w_1}{Av_1}, \underset{=w_2}{Av_2}, w_3\} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

Connections with Calculus.

Consider one-variable functions $f(x)$, and its derivative $f'(x) = \frac{df}{dx}$

We consider an restricted case $V = \text{Span}\{1, x, x^2\}$ the set of quadratic polynomials. Consider

$$p(x) = 6 - 4x + 3x^2$$

$$\frac{dp(x)}{dx} = 6 \cdot (\text{derivative of } 1)$$

$$- 4 \cdot (\text{derivative of } x) + 3 \cdot (\text{derivative of } x^2)$$

If we denote derivative by D, we see

$$D(ax^2 + bx + c) = a D(x^2) + b D(x) + c D(1)$$

$$= a(2x) + b(x) + c(0)$$

$$\in \text{Span}\{1, x\}$$

$V = \mathrm{Span}\{1, x, x^2\}$, $W = \{1, x\}$

The matrix for D w.r.t. the two bases

$$D\{1, x, x^2\} = \{1, x\} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The linear transformation D can be carried out by matrix-multiplication

Input $ax^2 + bx + c$ $\quad Au = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{pmatrix} b \\ 2a \end{pmatrix}$

output $2ax + b$

Now we consider integration

$$\int_0^x (\alpha x + \beta)\, dx = \beta x + \frac{1}{2}\alpha x^2 .$$

Input space basis $\{1, x\}$, output space basis $\{1, x, x^2\}$

$$D^+(1) = x, \quad D^+(x) = \frac{1}{2}x^2$$

$$D^+\{1, x\} = \{1, x, x^2\} \; \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{A^+}$$

matrix representation

Notice that $A^+A = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$AA^+ = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

A^+ is called the generalized inverse of A.

(Moore–Penrose)

<u>Example</u>. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$

Define a linear map $T: R^{2\times 2} \to R^{2\times 2}$

$T(X) = AX.$

Take the standard basis of $R^{2\times 2} = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, and compute

---

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & \cancel{0} \\ c & \cancel{0} \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Similarly for other basis matrices.

The matrix representation for T wrt the standard bases

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

Kernel and Image of a linear map.

let $T: V \to W$ be a linear map.

$$\ker(T) = \{ v \in V \mid T(v) = 0 \}$$

$$\operatorname{Im}(T) = \{ T(v), \mid v \in V \}$$

both are subspaces: $\ker(T) \subseteq V$, $\operatorname{Im}(T) \subseteq W$.

<u>Definition</u>. The rank of T, $r(T)$ ~~is~~ defined to be the dimension of $\operatorname{Im}(T)$: $r(T) = \dim(\operatorname{Im}(T))$.

Th. let $\{v_1, \cdots, v_n\}$, $\{w_1, \cdots, w_m\}$ be basis for V and W, respectively, and $A \in \mathbb{R}^{m \times n}$ is the matrix representation of $T: V \to W$. Then

$r(T) = r(A)$. Also let $Col(A) = \operatorname{Span}\{A_1, \cdots, A_r\}$ i.e., $A_1, \cdots, A_r$ are basis vectors of $Col(A)$, $N(A) = \operatorname{Span}\{B_1, \cdots, B_{n-r}\}$. Then

$$\left\{ \{w_1, \cdots, w_m\}A_1, \cdots \{w_1, \cdots, w_m\}A_r \right\}$$

forms a basis for $\mathrm{Im}(T)$, and

$$\left\{ \{v_1, \cdots, v_n\}B_1, \cdots, \{v_1, \cdots, v_n\}B_{n-r} \right\}$$

forms a basis for $\mathrm{Ker}(T)$.

Example (continue) $T(X) = AX$, and assume $a \neq 0$

but A is singular, since $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a \neq 0$, $\underbrace{d - ca^{-1}b}_{=\delta} = 0$

$$A_T \cancel{\neq A} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & & \end{array} \\ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$r = 2$, 1st, 2nd col's $\Longrightarrow$ independent col's

$$\mathrm{Col}(A_T) = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 0 \\ c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ 0 \\ c \end{pmatrix} \right\}$$

$$\mathrm{Im}(T) = \mathrm{Span}\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \right\}$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \right\}$$

For $\ker(T)$, the first two rows give

$$a x_1 + b x_3 = 0. \qquad x_1 = -\frac{b}{a} x_3$$

$$a x_2 + b x_4 = 0. \qquad x_2 = -\frac{b}{a} x_4.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{b}{a} & 0 \\ 0 & -\frac{b}{a} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$N(A_T) = \operatorname{Span}\left\{ \begin{pmatrix} -\frac{b}{a} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{b}{a} \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{Ker}(T) = \operatorname{Span}\left\{ \left(-\frac{b}{a}\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \left(-\frac{b}{a}\right)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \operatorname{Span}\left\{ \begin{pmatrix} -\frac{b}{a} & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \right\}$$