Lecture 22. Eigenvalues & Eigenvectors

If $A = I_n \in \mathbb{R}^{n\times n}$, then $\forall x \in \mathbb{R}^n$, $Ax = x$.

But for general A, Ax can be quite different from x.

Consider the Householder reflection

$$H = I_n - 2uu^T, \quad \|u\| = 1$$

Notice that the "mirror" is given by the hyperplane $S = \{x \mid u^T x = 0\}$

It's easy to check $\forall x \in S$, $Hx = x$

Also $\forall A \in \mathbb{R}^{n\times n}$, if $x = 0$, $Ax = x$. So we usually don't consider $x = 0$.

If we just ask for $Ax = x \Leftrightarrow (A-I)x = 0$.

for it to have nonzero sol, $A-I$ needs to be nonsingular. This is too restrictive, and we can ask for $x \neq 0$. s.t. Ax is parallel to x, i.e., there's a scalar λ, (which can be zero) s.t.

$$Ax = \lambda x, \quad x \neq 0. \quad A \in \mathbb{C}^{n \times n}$$

Such a pair (λ, x) is called an eigenvalue-eigenvector pair (or more concisely eigenpair) of A

This special vector x is streched, shrunk or reversed

Say $\lambda = 2$, $\lambda = \frac{1}{2}$, $\lambda = -1$.

~~Chach~~ Characterization of eigenpairs

$A \in \mathbb{R}^{n\times n}$, $\quad Ax = \lambda x$, $x \neq 0 \iff$

$(A - \lambda I_n) x = 0. \iff x \in N(A - \lambda I_n)$.

$x \neq 0$ implies $A - \lambda I_n$ is singular

$\iff \det(A - \lambda I_n) = 0.$

Example. $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$ $\quad \det\begin{pmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{pmatrix} = (\lambda - 1)(\lambda - \frac{1}{2})$

eigenvalues of A are $\lambda = 1$, $\lambda = -\frac{1}{2}$

eigenvectors $\quad (A - I) x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$

$(A - \frac{1}{2} I) x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Ax = \lambda x$, $\quad A^2 x = \lambda A x = \lambda^2 x$, $\cdots$, $\quad A^n x = \lambda^n x$.

Example transposition $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

eigenvalues of A $1, -1$.

$$A - I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A + I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Ax_1 = x_1, \quad Ax_2 = -x_2.$$

For small matrices, say 2x2, or 3x3 ones, we can find eigenpairs of A. by

1. find the roots of $\det(A - \lambda I) = 0$.
   say they are $\lambda_1, \ldots, \lambda_n$.
2. For each λ_i, solve for $(A - \lambda_i I) x_i = 0$.

Definition. The eigenspace corresponding to λ_i

$$= N(A-\lambda_i I) = \{\text{eigenvectors for } \lambda_i\} \cup \{0\}$$

This acknowledge the possibility that

$$\dim\big(N(A-\lambda_i I)\big) > 1.$$

and in this case λ_i is a multiple root of $\det(A-\lambda I)=0$.

Example $A = 2I$. eigenvalues of A $\lambda_1 = \lambda_2 = 2$.

$$N(A-2I) = R^2. \quad \dim\big(N(A-2I)\big) = 2 > 1.$$

$$\forall x \in R^2, \quad Ax = 2x.$$

Characteristic polynomial of $A \in R^{n\times n}$

$p_n(A) = \det(A - \lambda I)$.

$n=1$, $p_n(A)$ is a linear polynomial,

$n=2$ " quadratic "

For general n, $p_n(A)$ is degree n polynomial

with leading coefficient $(-1)^n$. $\left((-1)^n\lambda^n\right)$

Proof. $$\det\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn}-\lambda \end{pmatrix}$$

$$= (a_{11}-\lambda)\det\left(A(1,1) - \lambda I\right) + \cdots$$

or use the "Big formula" $\det(A - \lambda I)$

$$= \sum_{\sigma} \text{sign}(\sigma)\, \tilde{a}_{1\sigma(1)} \cdots \tilde{a}_{n\sigma(n)}$$

one term ~~for~~ $(a_{11}-\lambda)\cdots(a_{nn}-\lambda)$, other terms

missing at least one of the factors

Imaginary eigenvalues.

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ with } \theta = 90^\circ$$

rotation counterclock wise by 90°.

No vectors in $\mathbb{R}^2$ is an eigenvector.

$\det(A - \lambda I) = \lambda^2 + 1 = 0. \quad \lambda = \pm i.$

Complex eigenvectors. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = i\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Q is orth. $|\lambda| = 1$, Q is skew-symmetric $Q^T = -Q$.

eigenvalues are pure imaginary.

$\phi_n(\lambda)$ has n roots, counting multiplicity,

some roots can be complex.

1. If $A \in \mathbb{R}^{n\times n}$, and $\alpha + i\beta$ is an eigenvalue, then $\alpha - i\beta$ is also an eigenvalue

for a complex scalar $a \in \mathbb{C}$, denote its complex conjugate by $\overline{a}$

λ is an eigenvalue, $Ax = \lambda x$, $x \in \mathbb{C}^n$

$\Rightarrow \overline{A}\,\overline{x} = \overline{\lambda}\,\overline{x}$. (since $a_{i1}x_1 + \cdots + a_{in}x_n = \lambda x_i$

taking complex conjugate of both sides

$\overline{a_{i1}}\,\overline{x_1} + \cdots + \overline{a_{in}}\,\overline{x_n} = \overline{\lambda}\,\overline{x_i} \Rightarrow \overline{A}\,\overline{x} = \overline{\lambda}\,\overline{x}$)

since A is real, $A\overline{x} = \overline{\lambda}\,\overline{x}$, $x \neq 0 \Rightarrow \overline{x} \neq 0$.

2. A and A^T have the same list of eigenvalues

Proof. $\det(A^T-\lambda I)=\det(A-\lambda I)$

$p_{A^T}(\lambda)=p_A(\lambda)$.

3. Find the last row of $C=\begin{pmatrix}0&1&0\\0&0&1\\a&b&c\end{pmatrix}$

so that C has eigenvalues $1, 2, 3$.

$$\det(A-\lambda I)=\det\begin{pmatrix}-\lambda&1&0\\0&-\lambda&1\\a&b&c-\lambda\end{pmatrix}=\lambda^2(c-\lambda)+a+b\lambda$$

$$=(\lambda-1)(\lambda-2)(\lambda-3)$$

4. $A=\begin{pmatrix}B&C\\0&D\end{pmatrix}$, B and D are square matrices

eigenvalues of A is the combined lists of eigenvalues of B and D.

Proof. $\det(A-\lambda I)=\det(B-\lambda I)\det(D-\lambda I)$

5. A and B have the same ~~set~~ list of eigenvalues $\lambda_1, \cdots \lambda_n$, and the same list of eigenvectors $x_1, \cdots x_n$. Also $x_1, \cdots x_n$ are independent, then $A = B$.

Proof. Since $x_1, \cdots, x_n$ independent. $\forall x \in R^n$

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

Now
$$Ax = \alpha_1 Ax_1 + \cdots + \alpha_n Ax_n$$
$$= \alpha_1 \lambda_1 x_1 + \cdots + \alpha_n \lambda_n x_n$$
$$Bx = \alpha_1 Bx_1 + \cdots + \alpha_n Bx_n$$
$$= \alpha_1 \lambda_1 x_1 + \cdots + \alpha_n \lambda_n x_n$$

$\Rightarrow Ax = Bx \quad \forall x \in R^n \Rightarrow A = B.$

Conclusion: eigenvalues and (independent) eigenvectors determine the matrix!