Lecture 23, Spectral decomposition

Even for a real matrix, we may have to deal with possible complex eigenvalues

We first review complex numbers

$\mathbb{C}$ denotes the set of complex numbers, and $c \in \mathbb{C}$ can be written as $c = a + ib$, $i = \sqrt{-1}$, $a, b \in \mathbb{R}$

<u>Th.</u> A complex polynomial $p_n(x) = c_0 x^n + c_1 x^{n-1} + \dots + c_n = 0$.

with $c_0 \neq 0$, has n complex roots $x_1, x_2, \dots x_n$. (counting multiplicity)

<u>Arithmetic of complex numbers</u>: $z_1, z_2 \in \mathbb{C}$ $\begin{cases} z_1 = a + ib \\ z_2 = c + id \end{cases}$

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$$

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Complex <u>conjugate</u> $\bar{z} = \overline{a + ib} = a - ib$

absolute value: $|z|^2 = z\bar{z} = a^2 + b^2$, $z_1 / z_2 = \dfrac{z_1 \bar{z}_2}{|z_2|^2}$

<u>Euler formula</u> $e^{i\theta} = \cos\theta + i\sin\theta$, and plane rotation is the same as multiplication with $e^{i\theta}$: $\underbrace{z e^{i\theta}}$

$$z \to \begin{pmatrix} a \\ b \end{pmatrix}, \quad e^{i\theta} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Complex vectors $v \in \mathbb{C}^n$. $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $v_i \in \mathbb{C}$

length $\|v\|^2 = |v_1|^2 + \cdots + |v_n|^2 = \bar{v}^T v$

Dot-product $\bar{v}^T w = \bar{v}_1 w_1 + \cdots + \bar{v}_n w_n$

Now $\bar{v}^T w = \overline{\bar{w}^T v}$ (cf. for real vectors, $u^T v = v^T u$)

The length of $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is $\sqrt{2}$, of $\begin{pmatrix} 1+i \\ 0 \end{pmatrix}$ is $\sqrt{2}$.

Complex matrices $A \in \mathbb{C}^{m \times n}$, $(a_{ij}) = A$, $a_{ij} \in \mathbb{C}$

We introduce a notation $A^H = \bar{A}^T$; Hermitian transpose of A or complex conjugate transpose of.

Now dot product: $v^H w = \overline{w^H v}$

length $\|v\|^2 = v^H v$.

If $A \in \mathbb{R}^{n \times n}$, and $A = A^T$, A is symmetric

$A \in \mathbb{C}^{n \times n}$ and $A = A^H$, A is Hermitian

$S = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$ and its eigenvalues are $\lambda_1 = 8$, $\lambda_2 = -1$.

its corresponding eigenvectors

$$x_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}, \text{ and } x_1^H x_2 = 0.$$

Again $v^H w = 0$. we say $v \perp w$. <u>orthogonality</u>

Notice that if A is Hermitian, $a_{ii} \in \mathbb{R}$, $i = 1, \cdots, n$.

(*) $\mathbb{C}^n$ considered as complex vector space (i.e. scalars $\in \mathbb{C}$) is n-dimensional. $A \in \mathbb{C}^{m \times n}$

$$C(A) \oplus N(A^H) = \mathbb{C}^m; \text{ and } C(A)^\perp = N(A^H)$$

$$C(A^H) \oplus N(A) = \mathbb{C}^n.$$

$\mathbb{C}^n$ has standard basis $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Any $c \in \mathbb{R}^n$, $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

If $A^H x = 0$, $\forall y$ $y^H A^H x = (Ay)^H x = 0$.

$x \perp Ay$, $\forall y$.

Unitary matrices.

$\mathcal{U} \in \mathbb{C}^{n\times n}$ is unitary if $\mathcal{U}^H \mathcal{U} = I_n$. $\mathcal{U}^{-1} = \mathcal{U}^H$.

Th. for any $u_1 \in \mathbb{C}^n$, $\|u_1\| = 1$, we can find unitary matrix $\mathcal{U}$ s.t. $\mathcal{U} = [u_1, \mathcal{U}_1]$, i.e., we can expand u_1 into an orthonormal basis for $\mathbb{C}^n$.

Proof. We can use complex version of Householder

$$H = I - 2uu^H,\ \|u\| = 1 \text{ s.t.}$$

$$Hu_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow u_1 = H^H e_1 = He_1.$$

Notice that $H^H = H$, and $H^H H = I_n$.

Consider $A=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $A^H=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}=A$.

A is <u>Hermitian</u> Its characteristic polynomial is

$\det\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}-\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}\right)=\lambda^2-1=(\lambda-1)(\lambda+1)$

$\lambda_1=1,\ \lambda_2=-1$.

For $\lambda_1=1$. $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}\Rightarrow x_1=\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda_2=-1$ $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}-\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}=\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\Rightarrow x_2=\begin{pmatrix} 1 \\ i \end{pmatrix}$

In fact. $x_1^H x_2=(-i,1)\begin{pmatrix} 1 \\ i \end{pmatrix}=0$. $x_1\perp x_2$.

$U=\frac{1}{\sqrt{2}}(x_1,x_2)=\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$ is a unitary matrix.

$U^HU=\begin{pmatrix} u_1^H \\ u_2^H \end{pmatrix}(u_1,u_2)=\begin{pmatrix} u_1^Hu_1 & u_1^Hu_2 \\ u_2^Hu_1 & u_2^Hu_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

let $U_1=[u_1,u_2]$ $Au_1=\lambda_1u_1,\ Au_2=\lambda_2u_2$

$AU=A[u_1,u_2]=[Au_1,Au_2]=[\lambda_1u_1,\lambda_2u_2]$

$=[u_1,u_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}=U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\Rightarrow U^HAU=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

BTW, $-A=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is one of the four Pauli spin matrices used in quantum mechanics

Th. If $A \in \mathbb{C}^{n\times n}$ is Hermitian, then all eigenvalues of A are real numbers.

Proof. let (λ, x) be an eigenpair of A. i.e., $Ax = \lambda x$

Then $x^H A x = \lambda x^H x$. Now, $x^H x > 0$, and.

$(x^H A x)^H = x^H A^H x = x^H A x$, hence $x^H A x$ is real, then λ is real.

Th. If A is Hermitian, then there's a unitary matrix U, s.t. $U^H A U = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. If we let $U = [u_1, \cdots, u_n]$, then $Au_i = \lambda_i u_i$, i.e., (λ_i, u_i) is an eigenpair of A

Proof. let $Au_1 = \lambda_1 u_1$, $\|u_1\| = 1$. Expand u_1 into $U_1 = [u_1, *]$ a unitary matrix.

$AU_1 = [Au_1, *] = [\lambda_1 u_1, *]$

$$U_1^H A U_1 = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & * \end{array}\right], \text{ but}$$

$U_1^H A U_1$ is Hermitian, hence

$$U_1^H A U_1 = \left(\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_1 \end{array}\right),$$ and A_1 is also Hermi

The theorem is proved by induction.

<u>Corollary</u>. If $A \in \mathbb{R}^{n\times n}$ is symmetric, i.e., $A^T = A$, then there's ortho. Q, s.t. $Q^T A Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

<u>Proof</u>. $A u_1 = \lambda_1 u_1$, u_1 can be chosen as a real vector

in fact let $A(x+iy) = \lambda_1 (x+iy)$

$Ax = \lambda_1 x$, $Ay = \lambda_1 y$, $x+iy \neq 0 \Rightarrow x$ or $y \neq 0$.

Some properties

(HW) 1. A Hermitian, $u \in \mathbb{C}^n$, $\|u\|=1$.

$$u^H A u = \cancel{u^H Q \Lambda Q^H u}$$ where $A = U \Lambda U^H$ is the spectral decomposition

$$u^H U \Lambda U^H u$$

$$= v^H \Lambda v, \quad \text{where we define } v = U^H u.$$

notice that $\|v\| = \|U^H u\| = 1$

$$= \lambda_1 |v_1|^2 + \cdots + \lambda_n |v_n|^2, \quad \text{where } v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

We have the following characterization.

$$\lambda_{max}(A) = \max_{\|u\|=1} u^H A u, \quad \lambda_{min}(A) = \min_{\|u\|=1} u^H A u$$

where $\lambda_{max}(A)$, and $\lambda_{min}(A)$ are largest and smallest eigenvalues of A, respectively.

2. let $p(x)$ be a polynomial, then

$$p(A) = U \begin{pmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{pmatrix} U^H.$$

In particular, $A^k = U \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} U^H.$

(*) 3. let λ be an eigenvalue of $A=A^H$, and its mutiplicity as a root of the characteristic polynomial is k, then $\dim(N(A-\lambda I))=k$.

Notice that. $A-\lambda I=\mathcal{U}(\Lambda-\lambda I)\mathcal{U}^H$.

$\dim(N(A-\lambda I))=\dim(\Lambda-\lambda I)=k$.

(4) let $\lambda_1,\cdots,\lambda_n$ be the eigenvalues of A, then

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$\alpha\lambda_1+\beta,\cdots\alpha\lambda_n+\beta$ are the eigenvalues of $\alpha A+\beta I$.

Consider Householder $H=I-2uu^H$, $\|u\|=1$.

$\lambda(H)=1-2\lambda(uu^H)$

let $\mathcal{U}=[u,\mathcal{U}_1]$ be unitary, i.e. $\mathcal{U}^H\mathcal{U}=I$.

$uu^H\mathcal{U}=u[u^Hu,u^H\mathcal{U}_1]=u[1,0\cdots0]$

$=[u,u_1]\begin{bmatrix}1&&0\\&0&\\0&&\ddots\;0\end{bmatrix}$, $\mathcal{U}^H(uu^H)\mathcal{U}=\begin{pmatrix}1&&0\\&0&\\0&&0\end{pmatrix}$

$\lambda(H)=\{\underbrace{1,\cdots,1}_{n-1},-1\}$

(*) 5. Hermitian Matrices and Symmetric matrices

$A, B \in \mathbb{R}^{n\times n}$

$C = A + iB \in \mathbb{C}^{n\times n}$ be Hermitian, then

$$C^H = A^T - iB^T = A + iB \Rightarrow A = A^T, \cancel{B = B^T}$$
$$B = -B^T.$$

From $(A+iB)(x+iy) = \lambda(x+iy)$, $\lambda \in \mathbb{R}$, $x, y \in \mathbb{R}^n$

$$Ax - By = \lambda x, \quad Ay + Bx = \lambda y. \Leftrightarrow$$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} y \\ -x \end{pmatrix} = \lambda \begin{pmatrix} y \\ -x \end{pmatrix}$$

Notice that: $\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} y \\ -x \end{pmatrix} = x^T y - y^T x = 0.$

and $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}^T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, a symmetric matrix.