Lecture 24. Further Properties of Eigenvalues & Eigenvectors.

1. Similarity transformation.

recall when we consider $A \in \mathbb{R}^{n\times n}$ as a linear map

$$T: \mathbb{R}^n \to \mathbb{R}^n \quad T(u) = Au.$$

under standard basis, T has A as its matrix rep.

If we change the basis to $X \in \mathbb{R}^{n\times n}$, a nonsingular matrix, T's matrix rep wrt. X is given by

$$X^{-1}AX.$$

The transformation $A \longrightarrow X^{-1}AX$ is called a similarity transformation, and two matrices

A and B are similar, if there's a nonsingular X

s.t. $X^{-1}AX = B$.

Let's take a look at the eigenvalues / eigenvectors

~~be~~ of A and $X^{-1}AX$. Let.

$Ax = \lambda x$, then $(X^{-1}AX)(X^{-1}x) = \lambda(X^{-1}x)$

(λ, x) is an eigenpair of A iff.

$(\lambda, X^{-1}x)$ " $X^{-1}AX$.

Th. If A and B are similar, then they have the same list of eigenvalues, and their eigenvectors are related by $x \to X^{-1}x$.

In fact, $\det(X^{-1}AX - \lambda I) = \det X \det(A - \lambda I) \det X^{-1}$

here's another perspective, $= \det(A - \lambda I)$

Th. If A is triangular, then the eigenvalues are the diagonal entries of A.

Proof. $\det(A-\lambda I) = \det\begin{pmatrix} a_{11}-\lambda & & * \\ & \ddots & \\ 0 & & a_{nn}-\lambda \end{pmatrix}$

$= (a_{11}-\lambda)\cdots(a_{nn}-\lambda)$.

Question: Can we use similarity transformation to reduce A to upper triangular form, thus find all the eigenvalues of A?

Th. (Schur Decomposition) For $A \in \mathbb{C}^{n\times n}$, there is a unitary matrix U ($U^H U = I_n$) s.t. $U^H = U^{-1}$

$$\underbrace{U^H}_{U^{-1}} A U = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Proof. Similar to the proof of spectral decomposition.

Question: Is there a similarity transformation that reduce A to diagonal form?

Not necessarily, if $X^{-1}AX = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

A is said to be diagonalizable, and

$$p(A) = X \begin{pmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{pmatrix} X^{-1}$$ when p is a polynomial.

All matrices are not diagonalizable

1) For example, if the eigenvectors $x_1, \dots, x_n$, corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$ are linearly independent, then A is diagonalizable

2) If A's eigenvalues ~~λ_i are~~ λ_i are all distinct, A is diagonalizable

Question: what's the "simplest" form of a matrix $A \in \mathbb{C}^{n\times n}$ under similarity transformation.

Application of <u>diagonalizability</u>

Consider two functions $f: R \to R$, and $g: R \to R$

$$f(x) = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}}, \quad g(y) = \frac{b_{11}y + b_{12}}{b_{21}y + b_{22}}$$

$a_{ij}, b_{ij} \in R$.

both are a ~~rasions~~ ratios of two linear polynomials, Consider

$$f \circ g : R \to R. \qquad f \circ g(x) = f(g(x))$$

it turns out $f \circ g$ is also a ratios of two linear polynomials

$$f \circ g(x) = \frac{c_{11}x + c_{12}}{c_{21}x + c_{22}}$$

and

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

If we define $f^0(x) = x$, $f^1(x) = f(x)$, and

$f^n(x) = f(f^n(x))$, it's easy to see

$$f^n(x) = \frac{a_{11}^{(n)}x + a_{12}^{(n)}}{a_{21}^{(n)}x + a_{22}^{(n)}} \qquad \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{nn} \end{pmatrix}^n$$

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is diagonalizable

$$A = X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1}, \qquad A^n = X \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} X^{-1}$$

We can write $a_{ij}^{(n)}$, $i,j=1,2$, explicitly

Th. (Jordan Decomposition) For $A \in \mathbb{C}^{n\times n}$, there's X nonsingular, s.t. $X^{-1}AX = J$, where J is the Jordan canonical form. It's block diagonal and has diagonal blocks like

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}_{k\times k} \quad k \geq 2$$

It's easy to see, J_i has k eigenvalues $= \lambda_i$, but $\dim\left(N(J_i - \lambda_i I_k)\right) = k-1 < k$.

Th. (Cayley–Hamilton Th). Let $p_A(x)$ be the characteristic polynomial of $A \in \mathbb{C}^{n\times n}$, then

$$p_A(A) = 0$$

Proof. If A is diagonalizable, $A = X\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}X^{-1}$

$p_A(\lambda_i) = 0$, $\lambda_i = 1, \cdots, n$.

$$p_A(A) = X\begin{pmatrix}p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n)\end{pmatrix}X^{-1} = 0.$$

For general A, the result still holds #.

(by a limiting argument).

Consider $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(A-\lambda I_2)=\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$=(a-\lambda)(d-\lambda)-bc=a-(a+d)\lambda+(ad-bc)$$

Vieta th. says if x_1, x_2 are the two roots of $x^2+\alpha x+\beta=0$.

$$(x-x_1)(x-x_2)=x^2+\alpha x+\beta$$

$$x_1+x_2=-\alpha,\quad x_1x_2=\beta.$$

If λ_1, λ_2 are the eigenvalues of $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\lambda_1+\lambda_2=a+d,\qquad \lambda_1\lambda_2=ad-bc$$

$$=\operatorname{trace}(A)\qquad\qquad =\det A.$$

Trace of $A \in \mathbb{C}^{n \times n}$:

<u>Def.</u> $\mathrm{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$

the sum of the diagonal elements of A.

<u>Th.</u> $A, B \in \mathbb{C}^{n \times n}$, $\mathrm{trace}(AB) = \mathrm{trace}(BA)$

<u>proof.</u> direct verification

(i,i)-element of AB : $\sum_{k=1}^{n} a_{ik} b_{ki}$

$$\mathrm{trace}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

(k,k)-element of BA : $\sum_{i=1}^{n} b_{ki} a_{ik}$

$$\mathrm{trace}(BA) = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}$$

Th. $\mathrm{trace}(X^{-1}AX) = \mathrm{trace}(X)$

Proof. $\mathrm{trace}(X^{-1}AX) = \mathrm{trace}(AXX^{-1}) = A.$

Th. $\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i$, where λ_i are the eigenvalues of A, counting multiplicity.

Proof. Use Schur decomposition

$$U^{-1} A U = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}.$$

$$\operatorname{trace}(A) = \operatorname{trace}(U^{-1} A U) = \operatorname{trace}\begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i.$$

Th. $\det(A) = \prod_{i=1}^{n} \lambda_i \quad (= \lambda_1 \lambda_2 \cdots \lambda_n).$

Proof. Notice $\det S^{-1} = (\det S)^{-1}$, use Schur D.

$$\prod_{i=1}^{n} \lambda_i = \det \begin{pmatrix} \lambda_1 & & * \\ 0 & \ddots & \lambda_n \end{pmatrix} = \det(U^{-1} A U)$$

$$= \det U^{-1} \det A \det U = \det A.$$