Lecture 3

Matrices and their col spaces. $A \in \mathbb{R}^{m\times n}$

$$A = [a_1, a_2, a_3, a_4] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 4}.$$

difference matrix.

A $m\times n$. $A \in \mathbb{R}^{m\times n}$

matrix-vector multiplication.

Computation.

$$Ax = [a_1, a_2, a_3, a_4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix}$$

$$Ax = x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4.$$ linear comb. of the col's of A.

$$= x_1\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix}$$

All ~~comb~~ combinations of the col's $\{Ax,\ x \in \mathbb{R}^n\} \Rightarrow$ col space of A.

Some col's may not contribute anything new): they're combinations of earlier col's we ~~may~~ have already included

Example 1

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

each col gives a new dimension

the comb's fill R^3

$Ax = b.$ $\quad x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

(a_1, a_2, a_3) (linearly) independent

Theorem

$Ax = 0 \Rightarrow x = 0$

Example 2

$$\begin{array}{cc} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ A_2 = & \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 6 & 0 & 6 \end{bmatrix} \end{array}$$

$a_1 + a_2 = a_3$

two independent col's a_1 and a_2. Col space A is a plane

Example 3

$$A_3 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 15 & 20 \end{bmatrix}$$

$a_2 = 3a_1, \; a_3 = 4a_1.$

$a_1 \neq 0.$ independent.

Col space of A is a line $= \left\{ c \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \; c \in R \right\}$

$A \in R^{m \times n}; \; C(A) = \{ Ax : x \in R^n \}$

(col) Rank of a matrix: maximum number of independent columns of the matrix

Thinking ~~of~~ about the col space of A.

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ upper triangular.

all diagonal elements $\neq 0$. A_4 has 4 independent col's

$$A_5 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$ first tree col's independent.

$a_4 = a_1 - a_2 + a_3$

$C(A_4) \overset{?}{=} \mathbb{R}^4$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = (v_1 - v_2)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (v_2 - v_3)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (v_3 - v_4)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Notation $\quad \text{Span}\{a_1, \dots, a_n\} = \text{col}(A)$

the linear subspace $\underbrace{\text{spanned}}_{\text{spanned}}$ by $a_1, \dots, a_n$

$A \in \mathbb{R}^{m \times n}$, $\quad$ $Col(A)$ is a subspace of $\mathbb{R}^m$.

All possibilities for $\underline{\underline{m=3}}$.

1. A has 3 independent col's $\quad$ $Col(A) = \mathbb{R}^3$

2. $\quad$ 2 $\quad$ $Col(A) =$ a plane goes through $(0,0,0)$

3. $\quad$ 1

4. $\quad$ single ~~poin~~ vector $(0,0,0)$ $\quad$ $A = 0$.

$$1. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 2. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 3. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 4. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

How many col's of A are independent,

that number r is the "$\underbrace{\text{rank}}$" of A.

which are the first r independent col's of A?

They are a "basis" of col(A)

Matrices of rank one

<u>Theorem</u>: Every matrix of rank r is the sum of r matrices of rank one

$$A_6 = \begin{pmatrix} 1 & 3 & -2 \\ 4 & 12 & -8 \\ 2 & 6 & -4 \end{pmatrix} \quad \text{has rank} = 1$$

$a_2 = 3a_1,\ a_3 = -2a_1.$ $\quad C(A_6) = \text{Span}\{a_1\}$

$$= \text{Span}\left\{\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\right\}$$

$$= \left\{c\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} ;\ c \in \mathbb{R}\right\}$$

write A_6 row-wise $A_6 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

it's easy to see $b_2 = 4b_1$, $b_3 = 2b_1$.

row-space of A is also one-dimensional

More generally.

$$A = \begin{bmatrix} a & ma & pa \\ b & mb & pb \\ c & mc & pc \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$b_2 = \frac{b}{a} b_1, \quad b_3 = \frac{c}{a} b_1.$$

$\boxed{\text{row rank} = \text{column rank for all matrices!}}$

$$A_6 = \begin{pmatrix} 1 & 3 & -2 \\ 4 & 12 & -8 \\ 2 & 6 & -4 \end{pmatrix} \quad \underbrace{\text{rank} = 1}$$