Lecture 4

Matrix multiplication.

$$A \quad m\times n, \quad B \quad n\times p. \Rightarrow AB \quad m\times p.$$

several diff. perspective on AB.

col. way.

column j of AB

$$(AB)_j = AB_j. \qquad j\text{-th col of } AB = A \cdot j\text{-th col. of } B$$

$$B = [b_1, \cdots b_p] \qquad AB = [Ab_1, \cdots, Ab_p]$$

Example: $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$AB = \left[A\begin{bmatrix} 0 \\ 1 \end{bmatrix}, A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$ exchange the cols of A.

dot-product way

$$AB = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} [b_1, \cdots b_p] = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_p \\ a_m \cdot b_1 & \cdots & a_m \cdot b_p \end{bmatrix}$$

$$AB = \begin{bmatrix} (1,2)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} & (1,2)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (3,4)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} & (3,4)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

# of mutiplications each dot-product n.

total $= (m \cdot p) \cdot n = mnp$.

if $m = n = p$. $\Rightarrow m^3$. cubic in dimension

*. Strassen alg. $m=2 \Rightarrow 8. \rightarrow 7$ but with more additions

n^c $c = \log_2 7$ Deepmind uses RL to

recover and devise many fast matrix * alg.

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*. wall-clock time for matrix-multiply code.

Storage scheme, BLAS3.

$AB \neq BA$. A $m \times 1$. B $1 \times m$

AB $m \times m$. BA 1×1. Scalar!

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\underline{\underline{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$A\underline{\underline{B}} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

Associative law.

$$(AB)C = A(BC).$$

Suppose C has one col. $=c$ AB has col's

$$Ab_1, \cdots Ab_p$$

$$(AB)c = [Ab_1, \cdots Ab_p] \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

$$= c_1 Ab_1 + \cdots + c_p Ab_p.$$

$$Bc = c_1 b_1 + \cdots + c_p b_p$$

$$A(Bc) = A(c_1 b_1 + \cdots + c_p b_p) \overset{\text{linearity}}{=} c_1 Ab_1 + \cdots + c_p Ab_p$$

more cols. $C = [c_1, \ldots, c_q]$

$$(AB)C = [(AB)c_1, \ldots (AB)c_q]$$

$$A(BC) = A[Bc_1, \cdots, Bc_q] = [A(Bc_1), \ldots A(Bc_q)]$$

Distributive law $A(B+C) = AB + AC$

Partitioning of matrices $\Rightarrow$ block matrix.

$$A \quad m \times n. \qquad A = \begin{bmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & & \vdots \\ A_{q1} & \cdots & A_{qs} \end{bmatrix}$$

$$B \quad n \times p \qquad B = \begin{bmatrix} B_{11} & \cdots & B_{1t} \\ \vdots & & \vdots \\ B_{s1} & \cdots & B_{st} \end{bmatrix}$$

Compatible partition

$$(AB)_{ij} = \underbrace{A_{i1}}\,\underbrace{B_{1j}} + \cdots + A_{is}B_{sj}.$$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} [b_1, \cdots b_p] = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_p \\ \vdots & & \vdots \\ a_m \cdot b_1 & \cdots & a_m \cdot b_p \end{bmatrix}$$

$$A\,[b_1 \cdots b_p] = [Ab_1, \cdots Ab_p]$$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ \vdots \\ a_m B \end{bmatrix}$$

$$[\widetilde{a}_1, \cdots, \widetilde{a}_n] \begin{bmatrix} \widetilde{b}_1 \\ \vdots \\ \widetilde{b}_n \end{bmatrix} = \underbrace{\widetilde{a}_1 \widetilde{b}_1}_{\text{rank-one matrix.}} + \cdots + \widetilde{a}_n \widetilde{b}_n$$