Lecture 5

Four foundamental operations of linear algebra

1). vector addition, scalar-vector multiplication

$\forall u, v \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$.

$u + v \in \mathbb{R}^n, \quad \alpha u \in \mathbb{R}^n$

2) linear combinations of vectors

given $a_1, a_2, \cdots, a_n \in \mathbb{R}^m$

$x_1, x_2, \cdots, x_n \in \mathbb{R}$

$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \in \mathbb{R}^m$

3) Matrix-vector multiplications

block matrix / vector

$A = [a_1, \cdots, a_n] \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n$

$Ax = x_1 a_1 + \cdots + x_n a_n$

$A = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$b_i \in \mathbb{R}^{1 \times n}$

$Col(A) = \{ Ax : x \in \mathbb{R}^n \}$

$Ax = \begin{bmatrix} b_1 \cdot x \\ \vdots \\ b_m \cdot x \end{bmatrix}$

4)

4) Matrix-matrix multiplications

given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$

write $B = [b_1, \cdots b_p]$ where $b_i \in \mathbb{R}^{n\times 1}$
$i = 1, \cdots, p$

Define
$$AB = [Ab_1, \cdots, Ab_p] \in \mathbb{R}^{m\times p}$$

It can be verified, if $A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$

$A_i \in \mathbb{R}^{1\times n}$, then $AB = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} [b_1, \cdots, b_p]$

$$= \begin{bmatrix} A_1 \cdot b_1 & \cdots & A_1 \cdot b_p \\ \vdots & & \vdots \\ A_m \cdot b_1 & \cdots & A_m \cdot b_p \end{bmatrix}$$

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$AB = \left[A\begin{bmatrix} 0 \\ 1 \end{bmatrix}, A\begin{bmatrix} 1 \\ 0 \end{bmatrix}, A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1,2) \\ (3,4) \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1,2)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} & (1,2)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} & (1,2)\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ (3,4)\begin{pmatrix} 0 \\ 1 \end{pmatrix} & (3,4)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} & (3,4)\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{bmatrix}$$

The laws of matrix operations

1) linear combinations of matrices

$A \in R^{m\times n}$, $B \in R^{m\times n}$ $\alpha, \beta \in R$

$\alpha A + \beta B \in R^{m\times n}$

The set of $m\times n$ matrices forms a linear (vector) space

2) $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

$(AB)C = A(BC)$

However, $AB \neq BC$ in general

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ Is $(A+B)^2 = A^2 + 2AB + B^2$?

Block matrices

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]_{4\times 6} = \begin{bmatrix} I_2 & I_2 & I_2 \\ I_2 & I_2 & I_2 \end{bmatrix}$$

Matrix operations can be done a block at a time

$$A = \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \qquad B = \begin{array}{c} n_1 \\ n_2 \end{array} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{21}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Consider a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ A is nonsingular

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right)$$

Partitioning of matrices $\Rightarrow$ block matrix.

$$A \quad m \times n \qquad A = \begin{bmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & & \vdots \\ A_{q1} & \cdots & A_{qs} \end{bmatrix}$$

$$B \quad n \times p \qquad B = \begin{bmatrix} B_{11} & \cdots & B_{1t} \\ \vdots & & \vdots \\ B_{s1} & \cdots & B_{st} \end{bmatrix}$$

Compatible partition

$$(AB)_{ij} = \underbrace{A_{i1}}\,\underbrace{B_{1j}} + \cdots + A_{is} B_{sj}.$$

Four methods for matrix multiplications

① $$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} [b_1, \cdots b_p] = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_p \\ \vdots & & \vdots \\ a_m \cdot b_1 & \cdots & a_m \cdot b_p \end{bmatrix}$$

② $$A\,[b_1 \cdots b_p] = [Ab_1, \cdots Ab_p]$$

③ $$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ \vdots \\ a_m B \end{bmatrix}$$

④ $$[\widehat{a}_1, \cdots, \widehat{a}_n] \begin{bmatrix} \widehat{b}_1 \\ \vdots \\ \widehat{b}_n \end{bmatrix} = \underbrace{\widehat{a}_1 \widehat{b}_1}_{\text{rank\_one matrix.}} + \cdots + \widehat{a}_n \widehat{b}_n$$

2x2 Case: three possibilities: $Ax=b$

$A \in \mathbb{R}^{2\times2}$

1. One unique solution

$$A=\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} \qquad col(A)=\mathbb{R}^2; \ rank(A)=2$$

A is $\boxed{\text{nonsingular}}$/invertible   columns of A linearly ind.

2. No solution

$$A=\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \qquad col(A)\neq\mathbb{R}^2; \ rank(A)=1.$$

$b=\begin{pmatrix} 6 \\ 15 \end{pmatrix}$ columns of A dependent

$b \notin col(A)$ $\boxed{\text{A is singular}}$

3. Infinitely many solutions

$$b=\begin{pmatrix} 6 \\ 12 \end{pmatrix} \in col(A)$$

Since columns of A are dependent.

$\exists\, x \neq 0 \quad Ax=0$, also $b \in col(A)$

$\exists\, \tilde{x} \neq 0, \quad A\tilde{x}=b. \quad \forall \alpha \quad A(\tilde{x}+\underline{\underline{\alpha}}x)=b$

Mathematical Proofs.

1. For u, ~~and~~ v, and $w \in R^n$, show that.

$$\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

Proof. Recall the definition

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$$

$$v+w = \begin{pmatrix} v_1+w_1 \\ \vdots \\ v_n+w_n \end{pmatrix}$$

$$\langle u, v+w \rangle = \sum_{i=1}^{n} u_i (v_i+w_i)$$

$$= \sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} u_i w_i$$

$$= \langle u, v \rangle + \langle u, w \rangle$$

Chaucy—Schwarz inequality

$$|\langle u, v\rangle| \leqslant \|u\| \|v\|.$$

Proof. $\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \geqslant 0.$

Now we expand

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\|^2$$

$$= \left(\frac{u}{\|u\|} - \frac{v}{\|v\|}\right) \cdot \left(\frac{u}{\|u\|} - \frac{v}{\|v\|}\right)$$

$$= \frac{u \cdot u}{\|u\|^2} - 2 \frac{u \cdot v}{\|u\| \|v\|} + \frac{v \cdot v}{\|v\|^2}$$

$$= 2 - 2 \frac{u \cdot v}{\|u\| \|v\|} \geqslant 0 \quad = \langle u, v\rangle \leqslant \|u\| \|v\|.$$

Similarly we can prove $\langle u, v\rangle \geqslant - \|u\| \|v\|.$

$$\Rightarrow \quad |\langle u, v\rangle| \leqslant \|u\| \|v\|.$$

Th. Columns of A independent iff.

$Ax=0$ implies $x=0$.

Proof. "$\Longrightarrow$" prove by contradiction.

$\exists\ x\neq 0\quad Ax=0.\quad$ w.o.l.g. $x=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

and $x_1\neq 0$. $\quad Ax=0 \Leftrightarrow x_1a_1+\cdots+x_na_n=0$

$A=[a_1,\cdots,a_n]$

$$a_1=-\left(\frac{x_2}{x_1}\right)a_2-\cdots-\left(\frac{x_n}{x_1}\right)a_n$$

i.e., a_1 is a combination of $a_2,\cdots a_n$

$(\Leftrightarrow\ a_1\in \mathrm{Span}\{a_2,\cdots a_n\})$

"$\Longleftarrow$" prove by contradiction. Columns of A are NOT independent. w.o.l.g. a_1 is a combination of $a_2,\cdots,a_n$, $\quad a_1=\beta_2a_2+\cdots+\beta_na_n$

$$A\begin{bmatrix}1\\-\beta_2\\ \vdots\\-\beta_n\end{bmatrix}=a_1-\beta_2a_2-\cdots-\beta_na_n=0 \quad \begin{bmatrix}1\\-\beta_2\\ \vdots\\-\beta_n\end{bmatrix}\neq 0$$