lecture 6

①

solving linear systems

Big picture   triangular systems are easy to solve by backsubstition or forward-substition

~~Gaussian elimination~~ factorize A as

$$A = LU \quad \left(\begin{array}{c}\text{or more generally,} \\ PA = LU\end{array}\right)$$

where L is lower triangular, and U is upper triangular. Then

$$Ax = b \iff LUx = b.$$

let $c = Ux \Rightarrow \begin{cases} Lc = b \\ Ux = c \end{cases}$

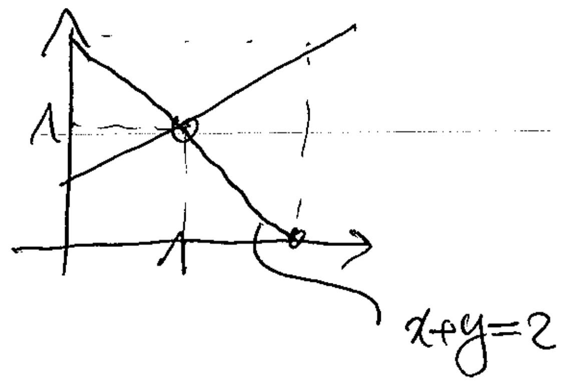
Geometry of Solving linear Systems

2×2 cases

three possibilities

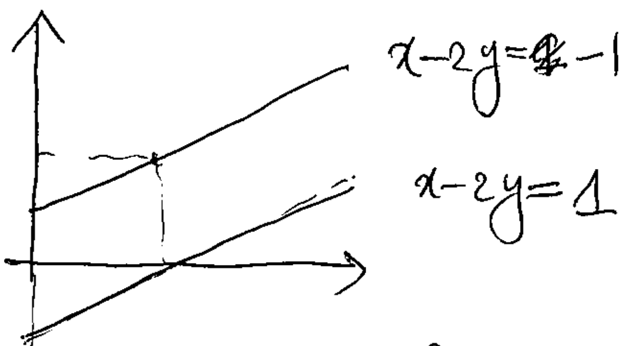
i) one solution

$$x - 2y = 1$$
$$x + y = 2$$



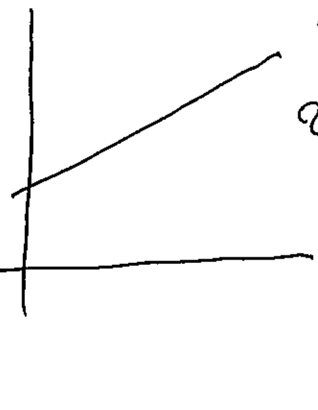
$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

ii) no solution



$$A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

iii) solution is a line



$$x - 2y = -1$$
$$2x - 4y = -2$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Back Substitution for Solving $Ux=c$

$$U=\begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & u_{22} \cdots & u_{2n} \\ 0 & & \ddots u_{nn} \end{pmatrix}$$

$$u_{11}x_1 + \cdots + u_{1n}x_n = c_1$$
$$u_{22}x_2 + \cdots + u_{2n}x_n = c_2$$
$$\vdots$$
$$u_{nn}x_n = c_n$$

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from the last equasion: if $u_{nn} \neq 0$, $x_n = c_n / u_{nn}$

next to the last eq:

$$u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = c_{n-1}$$

$$x_{n-1} = \left(c_{n-1} - u_{n-1,n}x_n\right) / u_{n-1,n-1}.$$

$$\vdots$$

$$x_1 = \left(c_1 - u_{12}x_2 - \cdots - u_{1n}x_n\right) / u_{11}.$$

Example $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 17 \\ 14 \end{pmatrix}$

Write $Ax=b$ row-wise

$$A=\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} \qquad A_i \in \mathbb{R}^{1\times n}$$

$$Ax=b \iff \begin{cases} A_1 \cdot x = b_1 \\ \vdots \\ A_n \cdot x = b_n \end{cases}$$

Substract $\alpha \times$ equation 1 from equation 2.

$$A_2 \cdot x - \alpha (A_1 \cdot x) = b_2 - \alpha b_1.$$

New eq 2: $\left(A_2 - \alpha A_1\right) \cdot x = b_2 - \alpha b_1.$

$$\tilde{A} = \begin{pmatrix} A_1 \\ A_2-\alpha A_1 \\ \vdots \\ A_n \end{pmatrix} \qquad \tilde{A}x=\tilde{b}$$

and

$$\tilde{b} = \begin{pmatrix} b_1 \\ b_2-\alpha b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad Ax=b$$

have the same solutions

<u>Matrix format</u>

$$E_{21}A = \begin{pmatrix} 1 & & & \\ -\alpha & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2-\alpha A_1 \\ \vdots \\ A_n \end{pmatrix}$$

$$E_{21}b = \begin{pmatrix} 1 & & & \\ -\alpha & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2-\alpha b_1 \\ \vdots \\ b_n \end{pmatrix}$$

order of elimination

Choose α s.t. the first element of $A_2 - \alpha A_1$ is zero.

$$a_{21} - \alpha a_{11} = 0 \Rightarrow \alpha = \frac{a_{21}}{a_{11}}$$

Example. $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix}$, $b = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix}$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_{21}A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{pmatrix}$$

$$E_{21} b = \begin{pmatrix} 19 \\ 17 \\ 50 \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad E_{31}(E_{21}A) = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{pmatrix}$$

$$E_{31}(E_{21}b) = \begin{pmatrix} 19 \\ 17 \\ 31 \end{pmatrix}$$

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix} \quad c = \begin{pmatrix} 19 \\ 17 \\ 14 \end{pmatrix}$$

zero-pivot.

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 5 & 13 \end{bmatrix} \equiv B.$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \underbrace{PB = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 13 \\ 0 & 0 & 8 \end{pmatrix}}_{U}$$

However, $A^* = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 3 & 17 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 4 \\ 0 & \underline{\underline{0}} & 6 \\ 0 & 0 & 13 \end{pmatrix} = U^*$

$\text{rank}(A^*) < 3$. not of full-rank. (or nonsingular)

<u>Th.</u> A triangular matrix U is nonsingular iff. all its diagonal elements are nonzero.

Example. For which three numbers a will elimination fail to give three pivot

$$A=\begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix} \to \begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{pmatrix} \to$$

$$\begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{pmatrix}$$

$a=0$, or $a=2$, or $a=4$.

Example which q makes this system singular and which t gives it infinitely many sol? Find the sol that has $z=1$.

$$x+4y-2z=1$$
$$x+7y-6z=6$$
$$3y+qz=t.$$

$$\begin{pmatrix} 1 & 4 & -2 \\ 1 & 7 & -6 \\ 0 & 3 & q \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ t \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & -4 \\ 0 & 3 & q \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ t \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & -4 \\ 0 & 0 & q+4 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ t-5 \end{pmatrix}$$

$q=-4 \Rightarrow$ singular

$t=5 \Rightarrow$ infinitely many sol

$$\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix} \underset{z=1}{\underline{\underline{1}}} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$