Lecture 7 — GE — Gaussian Elimination — A block-matrix perspective

1\. 1). $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

If $a_{11} \neq 0$. a_{11} is the "pivot", to eliminate the term $a_{21}x_1$ from equation ②,

Eq② $- \frac{a_{21}}{a_{11}}$ ⊗ equation ① : $l_{21} \equiv \frac{a_{21}}{a_{11}}$ "multiplier"

$$\left(a_{22} - \underbrace{\overset{l_{21}}{a_{21}a_{11}^{-1}}}\, a_{12}\right) x_2 = b_2 - \overset{l_{21}}{a_{21}a_{11}^{-1}} b$$

2) $a_{11} = 0$.

$0 \cdot x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

exchange eq ① and ②

$$\begin{cases} a_{21}x_1 + a_{22}x_2 = b_2 \\ a_{12}x_2 = b_1 \end{cases}$$

2\. If S is a nonsingular matrix, then

$Ax = b$ and $(SA)x = (Sb)$

have the same solution(s)

3.

1) Block diagonal matrices

$$\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \end{array} \begin{array}{c} n_1 \\ n_2 \end{array} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} CA_{11} & CA_{12} \\ DA_{21} & DA_{22} \end{pmatrix}$$

2) In particular, if $D = I_{n_2}$ then the last n_2 rows are not touched.

$$\begin{pmatrix} CA_{11} & CA_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

3)
$$\begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array} \begin{array}{c} p_1 \\ p_2 \end{array} \begin{pmatrix} E & \\ & F \end{pmatrix} = \begin{pmatrix} A_{11}E & A_{12}F \\ A_{21}E & A_{22}F \end{pmatrix}$$

4) Diagonal matrices $D = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{pmatrix} \equiv \mathrm{diag}(d_1, \dots, d_n)$

D is nonsingular iff $d_i \neq 0$, $i = 1, \dots, n$.

5) $\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} C^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$ <u>inverse</u> of a block diagonal matrix.

4. Solving $Ax=b$, $A \in \mathbb{R}^{n\times n}$, A nonsingular

write A row-wise, and define an elimination matrix

1) $$E_{21} = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ -l_{21} & 1 \end{matrix} & 0 \\ \hline 0 & I_{n-2} \end{array}\right)$$ block-diagonal

$$E_{21} A = \begin{array}{cc} 2 & n-2 \end{array}$$
$$E_{21} A = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ -l_{21} & 1 \end{pmatrix} & 0 \\ 0 & I_{n-2} \end{bmatrix} \begin{array}{c} 2 \\ n-2 \end{array} \begin{bmatrix} A_1 \\ A_2 \\ \hline A_3 \\ \vdots \\ A_n \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ -l_{21} & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \\ \hline A_3 \\ \vdots \\ A_n \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 \\ A_2 - l_{21} A_1 \\ \hline A_3 \\ \vdots \\ A_n \end{bmatrix}}_{\tilde{A}}$$

To make $(2,1)$ zero, $l_{21} = \dfrac{a_{21}}{a_{11}}$, multiplier

$$E_{21}^{-1} = \left(\begin{array}{c|c} \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} & 0 \\ \hline 0 & I_{n-2} \end{array}\right)$$

$$\tilde{b} = E_{21} b = \begin{bmatrix} b_1 \\ b_2 - l_{21} b_1 \\ \hline b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$Ax = b$

and $\tilde{A} x = \tilde{b}$

have the same sol because E_{21} is nonsingular

2) Write $\widetilde{A}$ row-wise $\widetilde{A} = \begin{pmatrix} \widetilde{A_1} \\ \vdots \\ \widetilde{A_n} \end{pmatrix}$

$$E_{31} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \\ -l_{31} & 0 & 1 & \\ \hline & 0 & & I_{n-3} \end{array}\right) \quad \text{block diagonal}$$

$$E_{31} \widetilde{A} = \left[\begin{array}{c} \widetilde{A_1} \\ \widetilde{A_2} \\ \widetilde{A_3} - l_{31} \widetilde{A_1} \\ \hline \widetilde{A_4} \\ \vdots \\ \widetilde{A_n} \end{array}\right]$$

To set $(3,1)$ element zero, $l_{31} = a_{31}/a_{11}$

Also $$E_{31} \widetilde{b} = \left[\begin{array}{c} \widetilde{b_1} \\ \widetilde{b_2} \\ \widetilde{b_3} - l_{31} \widetilde{b_1} \\ \hline \widetilde{b_4} \\ \vdots \\ \widetilde{b_n} \end{array}\right]$$

$$E_{31}^{-1} = \left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & 0 \\ l_{31} & 0 & 1 & \\ \hline & 0 & & I_{n-2} \end{array}\right) \quad \text{nonsingular}$$

3) Combining the previous two steps

$$E_{31}(E_{21}A)x = E_{31}(E_{21}b.)$$

lets take a look at

$$E_{31}E_{21} = \left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ -l_{31} & 0 & 1 & \\ \hline & & & I_{n-3} \end{array}\right) \left(\begin{array}{cc|c} 1 & & \\ -l_{21} & 1 & \\ \hline & & I_{n-2} \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ -l_{31} & 0 & 1 & \\ \hline & & & I_{n-3} \end{array}\right) \left(\begin{array}{ccc|c} 1 & & & \\ -l_{21} & 1 & & \\ 0 & 0 & 1 & \\ \hline & & & I_{n-3} \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & & & \\ -l_{21} & 1 & & \\ -l_{31} & 0 & 1 & \\ \hline & & & I_{n-3} \end{array}\right)$$

generalization:

$$\begin{pmatrix} 1 & & & 0 \\ -l_{21} & 1 & & \\ \vdots & 0 & \ddots & 1 \\ -l_{n1} & \cdots & & 1 \end{pmatrix}$$

$$E_{31}E_{21}A = \left[\begin{array}{c} A_1 \\ A_2 - l_{21}A_1 \\ A_3 - l_{31}A_1 \\ \hline A_4 \\ \vdots \\ A_n \end{array}\right], \quad E_{31}E_{21}b = \left[\begin{array}{c} b_1 \\ b_2 - l_{21}b_1 \\ b_3 - l_{31}b_1 \\ \hline b_4 \\ \vdots \\ b_n \end{array}\right]$$

5. Now it's natural to consider eliminating all the elements below a_{11} in the <u>first</u> colum

1) partition

$$A = \begin{array}{c c} & \begin{array}{c|c} 1 & n-1 \end{array} \\ \begin{array}{c} 1 \\ n-1 \end{array} & \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline a_1 & A_1 \end{array}\right) \end{array} \qquad a_{11} \neq 0.$$

$$E_1 = \left[\begin{array}{c|c} 1 & 0 \\ \hline -l_1 & I_{n-1} \end{array}\right]$$

$$E_1 A = \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline -l_1 a_{11} + a_1 & \underbrace{A_1 - l_1 u_1^T} \end{array}\right) \qquad \text{set the } (2,1) \text{ \underline{block} to zero}$$

$-l_1 a_{11} + a_1 = 0$, $l_1 = a_1 / a_{11}$ the vector of multipliers

$$l = \begin{pmatrix} a_{21}/a_{11} \\ a_{31}/a_{11} \\ \vdots \\ a_{n1}/a_{11} \end{pmatrix}, \quad E_1 = \left(\begin{array}{c|c} 1 & 0 \\ \hline -a_{21}/a_{11} & \\ \vdots & I_{n-1} \\ -a_{n1}/a_{11} & \end{array}\right)$$

$$E_1 b = \cancel{b_1} E \begin{pmatrix} b_1 \\ \hline \hat{b} \end{pmatrix} = \begin{pmatrix} b_1 \\ \hline \hat{b} - l_1 b_1 \end{pmatrix} = \begin{pmatrix} b_1 \\ \hline \tilde{b}_2 \end{pmatrix}$$

2) Consider the $(n-1)\times(n-1)$ block

$$\tilde{A}_2 \equiv A_1 - l u_1^T = \begin{array}{c|c} & \begin{array}{cc} 1 & n-2 \end{array} \\ \begin{array}{c} 1 \\ n-2 \end{array} & \left(\begin{array}{c|c} a_{22} & u_2^T \\ \hline a_2 & A_3 \end{array}\right) \end{array}$$

$$\tilde{E}_2 = \left(\begin{array}{c|c} 1 & \\ \hline -l_2 & I_{n-2} \end{array}\right)$$

$$\tilde{E}_2 \tilde{A}_2 = \left(\begin{array}{c|c} a_{22} & u_2^T \\ \hline a_2 - l_2 a_{22} & \underbrace{A_3 - l_2 u_2^T}_{\tilde{A}_3} \end{array}\right)$$

Set the $(2,1)$ to zero, $l_2 = a_2 / a_{22}$

$$\tilde{E}_2 \tilde{b}_2 = E_2 \left(\begin{array}{c} b_2 \\ \hline b_3 \end{array}\right) = \left(\begin{array}{c} b_2 \\ \hline b_3 - l_2 b_2 \end{array}\right)$$

Augmented matrix $[A, b]$

$$E\underbrace{[A, b]} = [\underbrace{EA}, \underbrace{Eb}]$$

3) We now combine the two steps

$$\begin{pmatrix} 1 & \\ & \widetilde{E}_2 \end{pmatrix} E_1 A = \left(\begin{array}{c|cc} a_{11} & \multicolumn{2}{c}{u_1^T} \\ \hline & a_{22} & u_2^T \\ \cline{2-3} 0 & 0 & \underbrace{\widetilde{A}_3}_{n-2} \end{array}\right) \Big\}{\scriptstyle n-2} \equiv \hat{A}_3$$

$$\boxed{(CD)^{-1} = D^{-1}C^{-1}} \Rightarrow \text{matrix inverses}$$

$$(*) \quad A = \underbrace{E_1^{-1} \begin{pmatrix} 1 & \\ & \widetilde{E}_2 \end{pmatrix}^{-1}} \underbrace{\hat{A}_3}$$

$$\begin{pmatrix} 1 & \\ & \widetilde{E}_2 \end{pmatrix} E_1 = \left(\begin{array}{c|c} 1 & \\ \hline & \underbrace{\begin{matrix} 1 & & 0 \\ -l_2 & \ddots & \\ & 0 & 1 \end{matrix}} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline -l_1 & I_{n-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline \underbrace{\begin{matrix} l_{21} \\ \widetilde{l}_1 - l_2 l_{21} \end{matrix}} & \begin{matrix} 1 & & \\ -l_2 & \ddots & 1 \end{matrix} \end{array}\right) \qquad l_1 = \left(\begin{array}{c} l_{21} \\ \hline \widetilde{l}_1 \end{array}\right)$$

$$\left(\begin{array}{c|c} 1 & \\ \hline -l_2 & I_{n-2} \end{array}\right) \begin{pmatrix} l_{21} \\ \widetilde{l}_1 \end{pmatrix} = \begin{pmatrix} l_{21} \\ \widetilde{l}_1 - l_2 l_{21} \end{pmatrix}$$

Now take a look at

$$E_1^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix}^{-1}$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline l_1 & I_{n-1} \end{array}\right)\left(\begin{array}{c|cc} 1 & & \\ \hline & 1 & \\ \cline{2-3} & l_2 & I_{n-2} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} 1 & & \\ \hline l_1 & 1 & \\ \cline{2-3} & l_2 & I_{n-2} \end{array}\right) \quad \text{simpler.}$$

Equation (*) implies

$$A = \left(\begin{array}{c|c|c} 1 & & \\ \hline l_1 & 1 & \\ \cline{2-3} & l_2 & I_{n-2} \end{array}\right)\left(\begin{array}{c|c|c} a_{11} & \multicolumn{2}{c}{u_1^T} \\ \hline & a_{22} & u_2^T \\ \cline{2-3} 0 & 0 & \tilde{A}_3 \end{array}\right)$$

a <u>partial</u> LU decomposition.

Proofs.

1. If $a_1, \dots, a_n$, contains a zero vector, then $a_1, \dots, a_n$ are dependent

2. If $a_1, \dots a_n$ are dependent, then $a_1, \dots, a_n, a_{n+1}, \dots a_m$ are also dependent

3. If columns of A are independent, then columns of $\begin{pmatrix} A \\ B \end{pmatrix}$ are also independent.

4. Let $A \in R^{m \times n}$, if $n > m$, columns of A are dependent.

5. Columns of a triangular matrix are independent iff all diagonal elements are nonzero.