# Lecture 8 PLU and Inverses

1\. 1) permutation: $(a_1, \dots a_n) \xrightarrow{\sigma} (a_{\sigma(1)} \dots a_{\sigma(n)})$

$$(1,2,3,4) \to (2,1,4,3). \quad n! \text{ permutation}$$

We want to reorder the rows or columns of a $A$.
which can be done by pre- or post-multiplication
of a permutation matrix. $P$: a reordering of the rows of an identity matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} A_3 \\ A_1 \\ A_2 \end{pmatrix}$$

$$(a_1, a_2, a_3) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (a_2, a_3, a_1)$$

$$(a_1, a_2, a_3) \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{P^T} = (a_3, a_1, a_2)$$

2) $PP^T = P^T P = I$, $P$ is a special orthogonal matrix

3) i) $A = (a_1, \dots, a_n)$, $A^T = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}$; ii) $A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$ $A^T = (A_1^T, \dots, A_n^T)$

4) a reordering of a reordered list is still a reordering of the original list. So the product of two permutation matrices $P_1 P_2$ is another permutation matrix., so is $P^T$ and $(P^{-1} = P^T)$

$$I_n = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \qquad e_i = (0 \cdots 0 \underset{\substack{\uparrow \\ i\text{-th position}}}{1} 0 \cdots 0)$$

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$P$ is a reordering of the rows of $I_n$., for book-keeping purposes we just need to use reordering of the list $(1, 2, \dots, n)$, and construct $P$ from the reordering of $(1, 2, \dots, n)$

5) Given two column vectors $u, v$ $\quad u \cdot v = (Pu) \cdot (Pv)$ if $P$ is permutation. Actually, the above is true if $P$ is orthogonal, i.e. $P^T P = I$.

$\Rightarrow$ The set of permutations (matrices) form a $\boxed{\text{group}}$ of cardinality $n!$

2. Th. If $A$ is nonsingular, then there's a permutation $P$ such that $PA = LU$ where $L$ is lower- and $U$ is upper-triangular

Proof. $A$ nonsingular, the first col of is nonzero

Use permutation matrix $P_1$, s.t. $P_1 A$ has a nonzero $(1,1)$ element

$$P_1 A = \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline a_1 & A_2 \end{array}\right), \quad a_{11} \neq 0.$$

let $l_1 = a_1 / a_{11}$, the multipliers $E_1 = \begin{pmatrix} 1 & 0 \\ -l_1 & I_{n-1} \end{pmatrix}$

$$E_1 P_1 A = \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & \underbrace{A_2 - l_1 u_1^T}_{\widehat{A_2}} \end{array}\right) \equiv \widehat{A}$$

$\widehat{A}$ is nonsingular $\Rightarrow \widehat{A_2}$ nonsingular

Use induction assumption, there's permutation

$$P_2 \widetilde{A}_2 = L_2 \mathcal{U}_2 \Rightarrow \widetilde{A}_2 = P_2^T L_2 \mathcal{U}_2$$

$$P_1 A = E_1^{-1} \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & P_2^T L_2 \mathcal{U}_2 \end{array}\right)$$

$$= \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_1 & I_{n-1} \end{array}\right) \left(\begin{array}{c|c} 1 & \\ \hline & P_2^T \end{array}\right)}\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_2 \end{array}\right) \overbrace{\underbrace{\left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & \mathcal{U}_2 \end{array}\right)}_{\text{upper-tri.}}}^{\mathcal{U}}$$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline l_1 & P_2^T \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_2 \end{array}\right) \mathcal{U}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & P_2^T \end{pmatrix} \left(\begin{array}{c|c} 1 & 0 \\ \hline P_2 l_1 & I_{n-1} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_2 \end{array}\right) \mathcal{U}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & P_2^T \end{pmatrix} \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline P_2 l_1 & \cancel{I_{n-1}} \, L_2 \end{array}\right)}_{L} \mathcal{U}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} P_1 A = L \mathcal{U}.$$

bookkeeping for permutation

$$\underbrace{\left(\begin{array}{ccc|c} 0&1&1&1\\ 2&4&1&2\\ 1&2&1&3\end{array}\right)}_{A \qquad b} \xrightarrow{P_1} \left(\begin{array}{ccc|c} 1&2&1&3\\ 2&4&1&2\\ 0&1&1&1\end{array}\right)$$

$$\xrightarrow{E_1} \left(\begin{array}{ccc|c} 1&2&1&3\\ 0&0&-1&2\\ 0&1&1&1\end{array}\right) \xrightarrow{P_2} \left(\begin{array}{ccc|c} 1&2&1&3\\ 0&1&1&1\\ 0&0&-1&\underline{\underline{2}}\end{array}\right)$$

$$U$$

$$E_1^{-1} = \begin{pmatrix} 1 & & \\ \boxed{2} & 1 & \\ \boxed{0} & 0 & 1\end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 1 & & \\ \boxed{0} & 1 & \\ \boxed{2} & 0 & 1\end{pmatrix}}_{L}$$

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$$P = \begin{pmatrix} 0&0&\underline{1}\\ 1&0&0\\ 0&1&0\end{pmatrix}$$

$$PA = \begin{pmatrix} 1&2&1\\ 0&1&1\\ 2&4&1\end{pmatrix} \xrightarrow{E_1} \underbrace{\begin{pmatrix} 1&2&1\\ 0&1&1\\ 0&0&-1\end{pmatrix}}_{U.}$$

$$E_1^{-1} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 2 & 0 & 1\end{pmatrix}$$

3. GE with <u>partial</u> pivoting

in the current column pick the element with the largest absolut value as the "pivot"

$l_1 = a_1 / a_{11}$. |each element of $l_1| \leq 1$.

4. GE with <u>complete</u> pivoting.

in the ~~whole~~ current ~~ts~~ submatrix, pick the element with the largest absolut value

$$PAQ = \left(\begin{array}{c|c} a_{11} & * \\ \hline * & * \end{array}\right) \quad \text{where } \underbrace{|a_{11}| = \max_{1 \leq i,j \leq n} |a_{ij}|}$$

$$PAQ(Q^T x) = (LU)\, Q^T x = Pb$$

$$x = Q(U^{-1}(L^{-1} Pb))$$

Transpose: properties

$$(A+B)^T = A^T + B^T, \quad (AB)^T = B^T A^T.$$

$$(A^{-1})^T = (A^T)^{-1}.$$

1) $B$ is single column $\quad Ax = x_1 a_1 + \cdots + x_n a_n$

$$x^T A^T = (x_1, \ldots, x_n) \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix} = x_1 a_1^T + \cdots + x_n a_n^T$$

2) $A^T (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I.$

Inverses. $A$ nonsingular $AX = I_n$. $\cancel{A^T}$.
$XA = I$. Then $X$ is called the inverse of $A$ and is denoted by $X = A^{-1}$.

Inverse is <u>unique</u>. $AX_1 = I$. $X_2 A = I \Rightarrow X_1 = X_2$

when does $A$ $\underbrace{\text{has an inverse}}_{\text{nonsingular}}$?

1) $(AB)^{-1} = B^{-1}A^{-1}$. (verify the definition).

2) $A$ has indepent columns $\Leftrightarrow$ $A$ has inverse

$PA = LU$. $L$ has 1 as its diagonals

$U$ has nonzero diagonals.

$P$ is permutation

3) Triangular matrices are nonsingular iff all its diagonals are nonzero.

~~Notes~~ ($n \times n$)

$\Rightarrow$ The set of nonsingular matrix forms a $\boxed{\text{group}}$ under ~~the~~ matrix multiplication, and the set of permutation matrices is a finite subgroup with cardinality $n!$

Group theoretic proof.

the existence of $X$ s.t. $AX=I$ is guaranteed by Gaussian elimination.

Now we show $X$ has independent columns

In fact, $Xx=0 \Rightarrow A\underbrace{Xx}_{=0}=x \Rightarrow x=0$.

Then by GE, there exists $Y$ s.t. $XY=I$.

$A(XY)=AI. \Rightarrow \underbrace{(AX)}_{I}Y=A \Rightarrow Y=A$. i.e. $X$ is the inverse of $A$.

Another similar proof. We show $A^T$ has independent columns $AX=I \Rightarrow X^TA^T=I$. $A^Tx=0 \Rightarrow$ $0=X^TA^Tx=x \Rightarrow x=0$.

By GE, there's $Y$, s.t. $A^TY=I$ or $Y^TA=I$.

$\Rightarrow Y^T=X$, i.e. $X$ is the inverse of $A$.

# Computation of Inverses

$$AX = I = (e_1, e_2, \dots, e_n)$$

1) Compute the PLU decomposition of A.

$$PA = LU$$

2) for $i = 1, \dots, n$. Compute the solution of

$$Ax_i = e_i$$

$$PLUx_i = e_i \qquad LUx_i = P^T e_i$$

i). $L c_i = P^T e_i$ ii) $U x_i = c_i$

3) $X = [x_1, \dots, x_n] \in R^{n \times n}$.

$\Rightarrow$ Notice that PLU decomposition is done once for each new rhs $e_i$, just two triangular solves!