# lecture 9. PLU. (updated).

Review of Gaussian elimination.

Th. $\forall b \in \mathbb{R}^n$, $Ax=b$ has a solution iff $A$ has independent columns. Then $\mathrm{col}(A)=\mathbb{R}^n$, $A$ is invertible and the solution of $Ax=b$ is unique

GE:

$$\left(\begin{array}{c|cc} I_{n-2} & & \\ \hline & 1 & \\ & -l_{n-1} & 1 \end{array}\right) \cdots \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & -l_2 & I_{n-2} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline -l_1 & I_{n-1} \end{array}\right) A = \overset{\equiv \mathcal{U}.}{\left(\begin{array}{c|c|c|c} a_{11} & \multicolumn{3}{c}{u_1^T} \\ \hline & a_{22} & \multicolumn{2}{c}{u_2^T} \\ \hline & & \ddots & \cdots \\ \hline & & & \begin{matrix} a_{n-1\,n-1} & u_{n-1} \\ 0 & a_{nn} \end{matrix} \end{array}\right)}$$

$$A = \left(\begin{array}{c|c} 1 & \\ \hline l_1 & I_{n-1} \end{array}\right) \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & l_2 & I_{n-2} \end{array}\right) \cdots \left(\begin{array}{c|cc} I_{n-2} & & \\ \hline & 1 & \\ & l_{n-1} & 1 \end{array}\right) \mathcal{U} \equiv L\mathcal{U}$$

A matrix product representation of triangular matrices

let $I_n = (e_1, e_2, \dots, e_n) = \begin{pmatrix} e_1^T \\ \vdots \\ e_n^T \end{pmatrix} \in R^{n\times n}$

lower triangular $L = (l_1, l_2, \dots, l_n)$. $l_{ii} \neq 0$.

$$L = L I_n = l_1 e_1^T + \dots + l_n e_n^T$$

$$I_n = I_n I_n = e_1 e_1^T + \dots + e_n e_n^T$$

$$L = I + (l_1 - e_1) e_1^T + \dots + (l_n - e_n) e_n^T.$$

$$= I + \widetilde{l}_1 e_1^T + \dots + \widetilde{l}_n e_n^T, \quad \widetilde{l}_i = l_i - e_i$$

$$= (I + \widetilde{l}_1 e_1^T)(I + \widetilde{l}_2 e_2^T)\cdots(I + \widetilde{l}_n e_n^T),$$ notice that. $e_j^T l_i = 0$, $j < i$

Also. $(I + \widetilde{l}_i e_i^T)^{-1} = I - \frac{1}{l_{ii}} \widetilde{l}_i e_i^T$

$$L^{-1} = \left(I - \frac{1}{l_{nn}} \widetilde{l}_n e_n^T\right) \cdots \left(I - \frac{1}{l_{11}} \widetilde{l}_1 e_1^T\right)$$

if $l_{ii} = 1$, $\widetilde{l}_n = 0$,

$$\begin{cases} L = (I + \widetilde{l}_1 e_1^T)\cdots(I + \widetilde{l}_{n-1} e_{n-1}^T) \\ L^{-1} = (I - \widetilde{l}_{n-1} e_{n-1}^T)\cdots(I - \widetilde{l}_1 e_1^T) \end{cases}$$

# Lecture 8 PLU and Inverses

1\. 1) permutation: $(a_1, \dots a_n) \xrightarrow{\sigma} (a_{\sigma(1)} \dots a_{\sigma(n)})$

$(1, 2, 3, 4) \to (2, 1, 4, 3)$. $n!$ permutation

We want to reorder the rows or columns of a $A$. which can be done by pre- or post-multiplication of a permutation matrix. $P$: a reordering of the rows of an identity matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} A_3 \\ A_1 \\ A_2 \end{pmatrix}$$

$$(a_1, a_2, a_3) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (a_2, a_3, a_1)$$

$$(a_1, a_2, a_3) \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{P^T} = (a_3, a_1, a_2)$$

2) $PP^T = P^T P = I$, $P$ is a special orthogonal matrix

3) i) $A = (a_1, \dots, a_n)$, $A^T = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}$; ii) $A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$ $A^T = (A_1^T, \dots, A_n^T)$

4) a reordering of a reordered list is still a reordering of the original list. So the product of two permutation matrices $P_1 P_2$ is another permutation matrix., So is $P^T$ and $(P^{-1} = P^T$

$$I_n = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \qquad e_i = (0 \cdots 0 \underset{\uparrow}{1} 0 \cdots 0)$$

i-th position

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P is a reordering of the rows of $I_n$.) for book-keeping purposes we just need to use reordering of the list $(1, 2, \dots, n)$, and construct P from the reordering of $(1, 2, \dots, n)$

5) Given two column vectors $u, v$ $\quad u \cdot v = (Pu) \cdot (Pv)$ if P is permutation. Actually, the above is true if P is orthogonal, i.e. $P^T P = I$.

$\Rightarrow$ The set of permutations (matrices) form a $\boxed{\text{group}}$ of $\overset{\text{Cardinal}}{n!}$

2. **Th** If $A$ is nonsingular, then there's a permutasion $P$ such that $PA=LU$ where $L$ is lower- and $U$ is upper-triangular

Proof. $A$ nonsingular, the first col of is nonzero

Use permutasion matrix $P_1$, s.t. $P_1A$ has a nonzero $(1,1)$ element

$$P_1A = \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline a_1 & A_2 \end{array}\right), \quad a_{11} \neq 0.$$

let $l_1 = a_1/a_{11}$, the multipliers $E_1 = \begin{pmatrix} 1 & 0 \\ -l_1 & I_{n-1} \end{pmatrix}$

$$E_1P_1A = \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & \underbrace{A_2 - l_1u_1^T}_{\widehat{A_2}} \end{array}\right) \equiv \widehat{A}$$

$\widehat{A}$ is nonsingular $\Rightarrow \widehat{A_2}$ nonsingular

Use induction assumption, there's permutation

$$P_2 \widehat{A}_2 = L_2 U_2 \Rightarrow \widehat{A}_2 = P_2^T L_2 U_2$$

$$P_1 A = E_1^{-1} \left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & P_2^T L_2 U_2 \end{array}\right)$$

$$= \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_1 & I_{n-1} \end{array}\right) \left(\begin{array}{c|c} 1 & \\ \hline & P_2^T \end{array}\right)} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_2 \end{array}\right) \underbrace{\overset{U}{\left(\begin{array}{c|c} a_{11} & u_1^T \\ \hline 0 & U_2 \end{array}\right)}}_{\text{upper-tri.}}$$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline l_1 & P_2^T \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_2 \end{array}\right) U$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & P_2^T \end{pmatrix} \left(\begin{array}{c|c} 1 & 0 \\ \hline P_2 l_1 & I_{n-1} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_2 \end{array}\right) U$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & P_2^T \end{pmatrix} \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline P_2 l_1 & \cancel{I_{n-1}}\, L_2 \end{array}\right)}_{L} U$$

$$\begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} P_1 A = L U.$$

## Example

↙ book-keeping for permutation

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{array}\right) \xrightarrow{P_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$A \quad p.$

$$\xrightarrow{E_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{P_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & \underline{\underline{2}} \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{matrix}}_{U.}$$

$$E_1^{-1} = \begin{pmatrix} 1 & & \\ \boxed{2} & 1 & \\ \boxed{0} & 0 & 1 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 1 & & \\ \boxed{0} & 1 & \\ \boxed{2} & 0 & 1 \end{pmatrix}}_{L}$$

$$P = \begin{pmatrix} 0 & 0 & \underline{1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$PA = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 1 \end{pmatrix} \xrightarrow{E_1} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}}_{U.}$$

$$E_1^{-1} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 2 & 0 & 1 \end{pmatrix}$$

3. GE with partial pivoting

in the current column pick the element with the largest absolute value as the "pivot"

$l_1 = a_1 / a_{11}$ $\quad |$each element of $l_1| \leqslant 1$.

4. GE with complete pivoting.

in the current submatrix, pick the element with the largest absolute value

$$PAQ = \left(\begin{array}{c|c} a_{11} & * \\ \hline * & * \end{array}\right) \quad \text{where } \underbrace{|a_{11}| = \max_{1 \leqslant i,j \leqslant n} |a_{ij}|}$$

$$PAQ\,(Q^T x) = (LU)\,Q^T x = Pb$$

$$x = Q(U^{-1}(L^{-1}Pb))$$

For GE with partial pivoting.

$$L^{-1} = (I - \widetilde{l}_{n-1} e_{n-1}^T) P_{n-1} \cdots (I - \widetilde{l}_2 e_2^T) P_2 (I - \widetilde{l}_1 e_1^T) P_1$$

notice that $P_i$ only operates on rows $i, i+1, \dots n$.

therefore $e_{i-1}^T P_i^T = e_{i-1}^T$, and.

$$P_i (I - \widetilde{l}_{i-1} e_{i-1}^T) = (I - (P_i \widetilde{l}_{i-1}) e_{i-1}^T) P_i$$

Hence $L^{-1} = (I - \widetilde{l}_{n-1} e_{n-1}^T)(I - \underbrace{P_{n-1}}_{} l_{n-2} e_{n-2}^T) \cdots$

$$\cdots (I - \underbrace{P_{n-1} \cdots P_{i+1}}_{} \widetilde{l}_i e_i^T) \cdots (I - \underbrace{P_{n-1} \cdots P_2}_{} \widetilde{l}_1 e_1^T) \underbrace{P_{n-1} \cdots P_1}_{}$$

$$L = \underbrace{P_1 \cdots P_{n-1}}_{} (I + P_{n-1} \cdots P_2 \widetilde{l}_1 e_1^T) \cdots (I + \widetilde{l}_{n-1} e_{n-1}^T)$$

5. Transpose: properties

$$(A+B)^T = A^T + B^T, \quad (AB)^T = B^T A^T.$$

$$(A^{-1})^T = (A^T)^{-1}, \quad \boxed{(Ax)^T y = x^T A^T y}$$

1) B is single column $\quad Ax = x_1 a_1 + \dots + x_n a_n$

$$x^T A^T = (x_1, \dots, x_n) \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix} = x_1 a_1^T + \dots + x_n a_n^T$$

2) $A^T (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I.$

6 Inverses A nonsingular $AX = I_n$. $\cancel{A^T X = I}$.

$XA = I$. Then $X$ is called the inverse of $A$ and is denoted by $X = A^{-1}$.

Inverse is unique. $AX_1 = I$. $X_2 A = I \Rightarrow X_1 = X_2$