Lecture 01

Introduction to Linear Algebra and Data Science

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Part I Data Science and Linear Algebra

Why Data are Important

Data are the new "oil/petroleum" and "power"!

Petroleum/power can drive cars, airplanes, cellphones, …

"Physical" economy

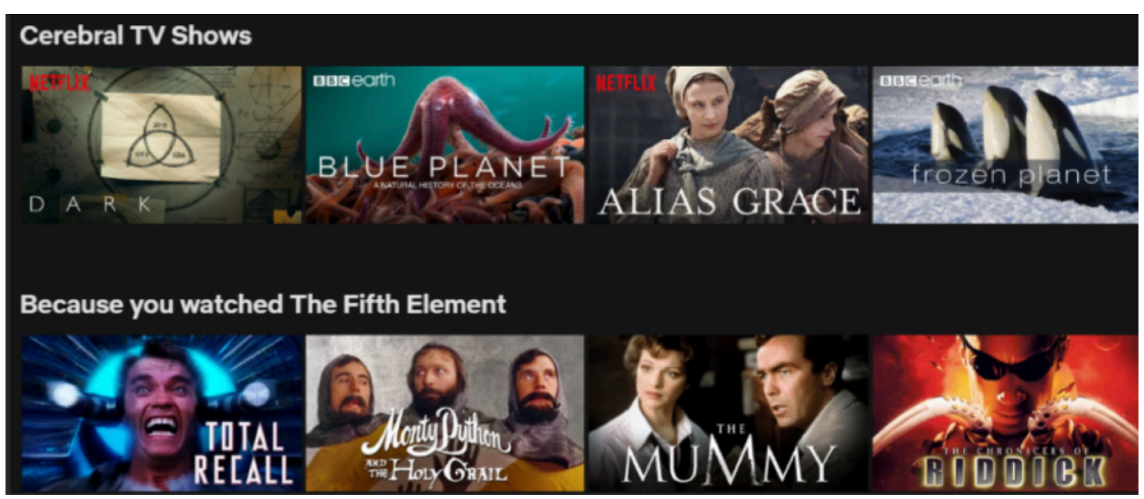
Data can drive AI and decision making,

Digital economy

What are “Data”?

Images are “data”

Stock prices are data



Movies are “data”

Galaxy observations are data

First Challenge: How to “Express” them?

$$\begin{bmatrix} 2.03 \\ 0.87 \\ 0.65 \\ 0.96 \\ \ldots \end{bmatrix}$$

Figure? Human likes!

vector! Computer likes!

First Challenge: How to "Express" them?

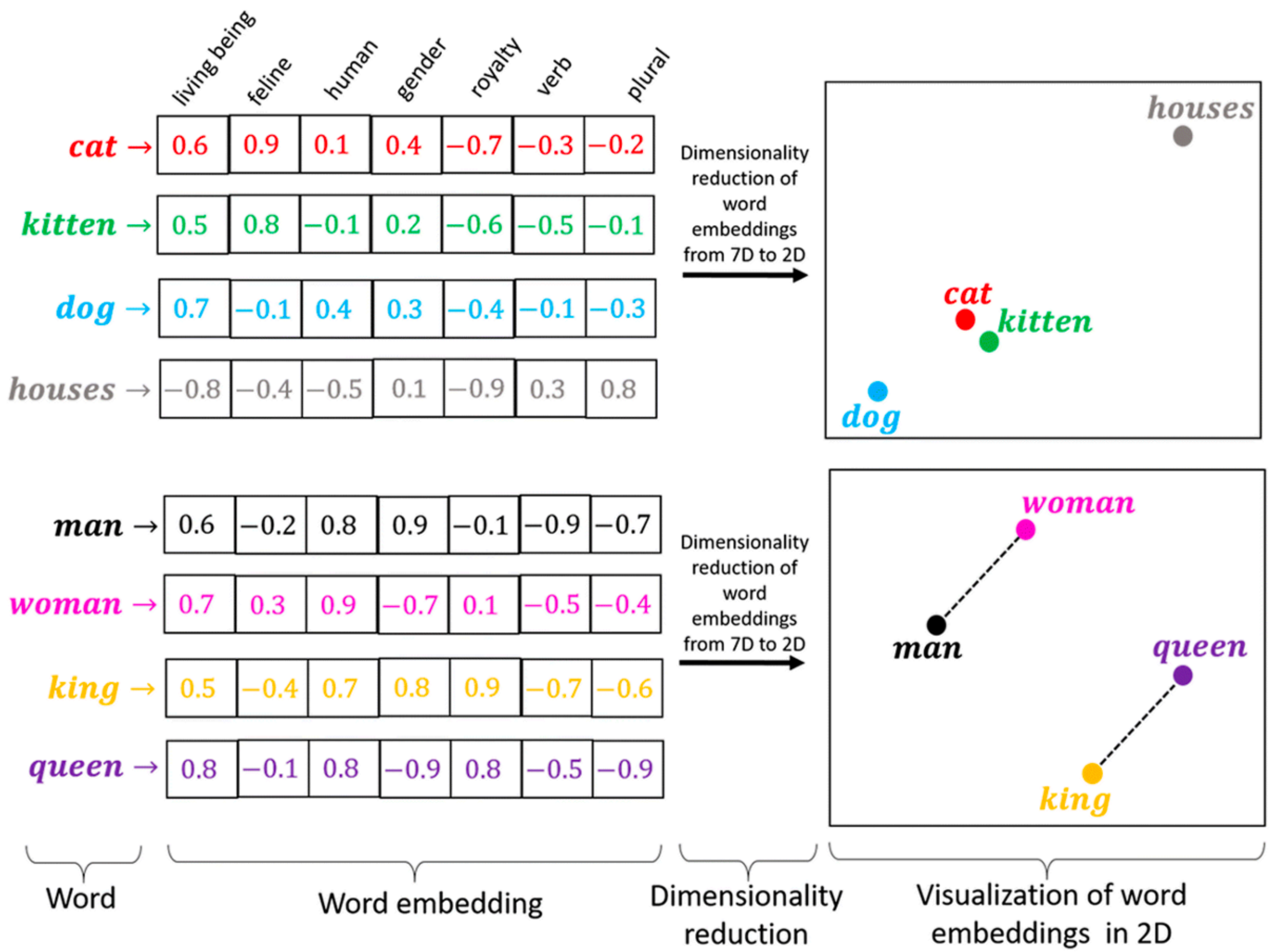
157	153	174	168	150	152	129	151	172	161	155	156
155	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	106	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234

Figure? Human likes!

Matrix! Computer likes!

Advanced Example 1 Words represented as vectors

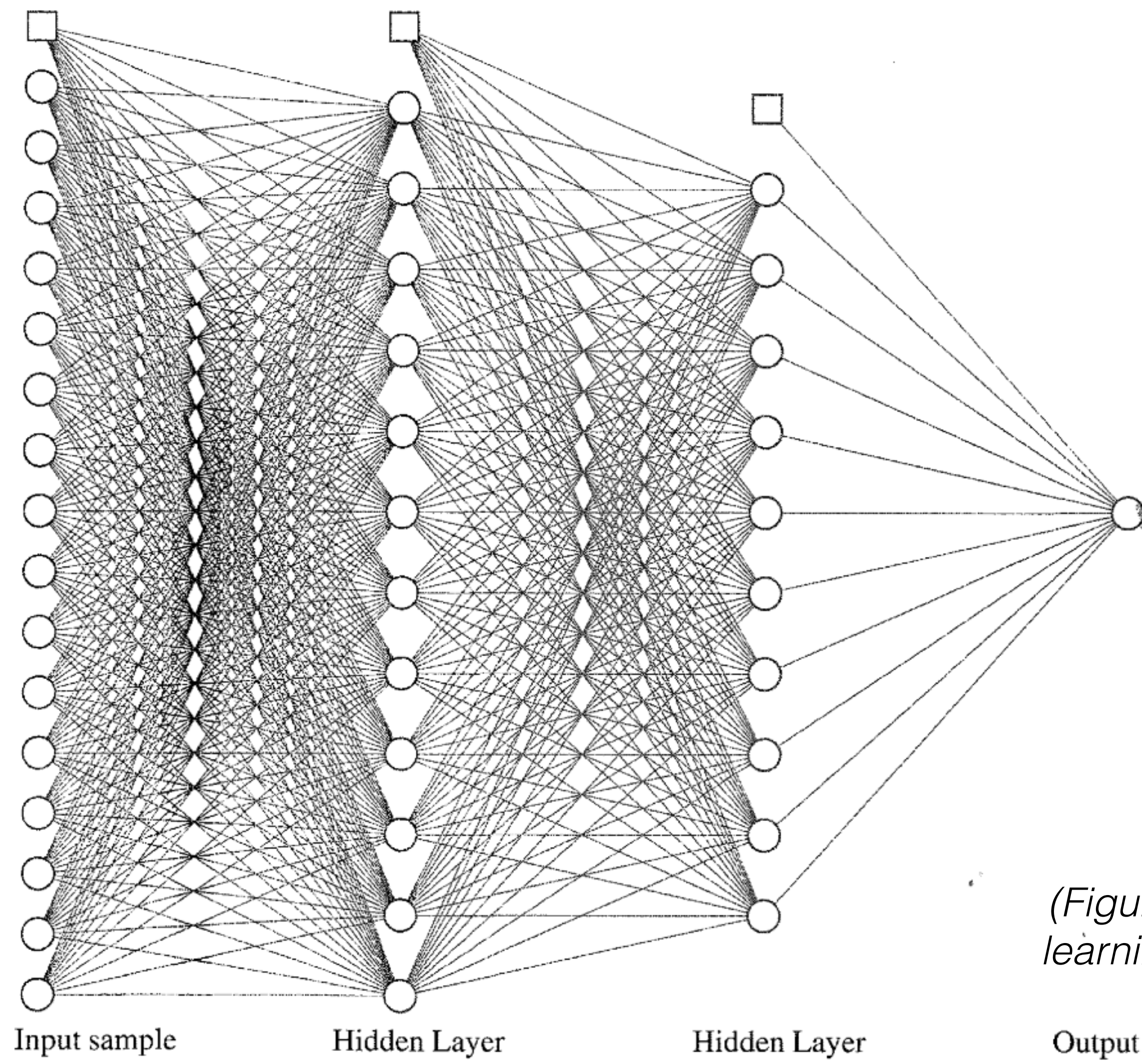
Word Embedding: each word can be represented by a vector



(Figure resource: https://medium.com/@hari4om/word-embedding-d816f643140)

Advanced Example 2 Weights of neural networks

Convolutional Neural Networks (CNNs)



(Figure resource: linear algebra and learning from data by Gilbert Strang)

- "Neurons" mimic "human brain neurons"
- Connections mimic interactions between neurons
- Connections form a matrix

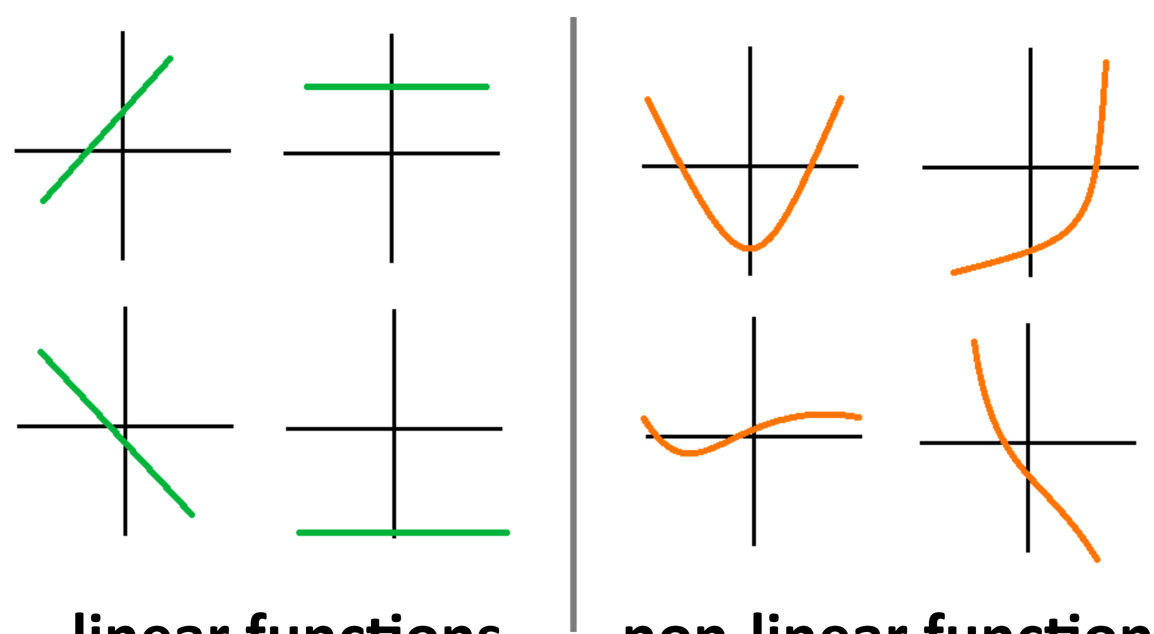
What is Linear Algebra?

Algebra: Mathematical representation of numbers and operations

X, Y, x, y, A, B, …

addition +
subtraction -
multiplication ×
inner product $\langle \cdot , \cdot \rangle$ …

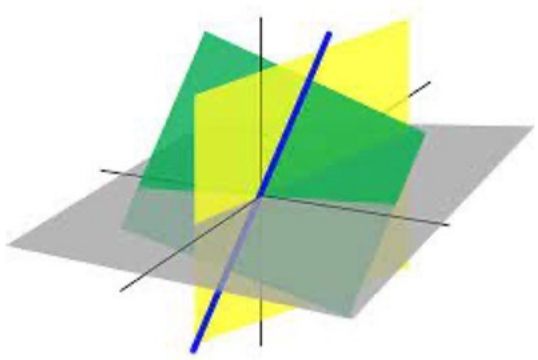
Linearity:



2-D representation

linear functions **non-linear functions**

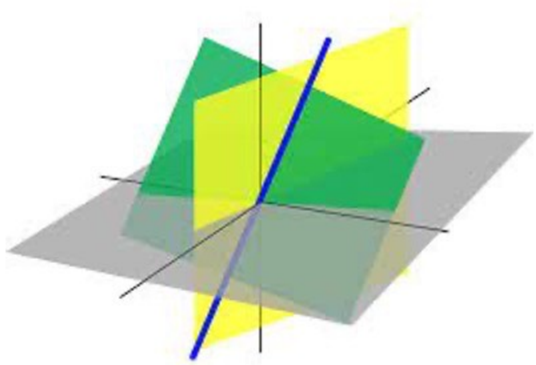
What is Linear Algebra?



Mathematically, vectors, matrices;

vector spaces, linear transformations…

What is Linear Algebra?



Mathematically, **vector spaces** and linear transformations

Applications

- **Machine learning and data science**
- Computer vision and graphics
- Graph theory
- Control theory
- Cryptography
- Fractals and chaos
- Energy systems
- Network systems
- Genetics
- Etc. …

Practically, it can be applied to **any** problems with vector, matrix-type data, and linear models

Why Need Linear Algebra? A1: Fundamental

Quora

Search for questions, people, and topics

What exactly is linear algebra? Why do we need it?

Why study linear algebra?

Ask Q

Asked 9 years, 8 months ago Modified 3 years, 4 months ago Viewed 133k times

128

Simply as the title says. I've done some research, but still haven't arrived at an answer I am satisfied with. I know the answer varies in different fields, but in general, why would someone study linear algebra?

https://math.stackexchange.com/questions/256682/why-study-linear-algebra

Linear algebra is beyond important, it is fundamental to so many fields that I cannot count them all.

Having studied Engineering, I can tell you that Linear Algebra is fundamental and an extremely powerful tool in **every single** discipline of Engineering.

3D computer graphics? Linear algebra. Quantum mechanics? Linear algebra. Weather forecast models? Linear algebra.

Study it if you are into economics, computer graphics, physics, chemistry, statistics or anything quantitative (in today's world, that's everything).

Why Learning Linear Algebra is Important?

- It can be applied to **many** problems
- **Easy** to model, analyze, and compute (not an easy subject)
- Foundations of more advanced and complex methods

Linear Models	Nonlinear Models
Less accurate	More accurate
Easy	Hard

This Course

Not a pure math course

For instance, we will not dive into the "axioms" when studying vector spaces

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More data science problems will be demonstrated to motivate the concepts (compared to MAT2040)

For instance, least squares, graph matrices, Searching, epidemics etc. (will cover as much as possible)

Learning Methodology

Reading Slides are not enough for learning **math** …

Study Strategy

Try to be an instructor yourself. Explain what you learn to yourself & classmates. Ask yourself questions like below and write down answers.
—What do I learn in this lecture?
—What do I learn in the past month?

Work on Proofs: Many linear algebra courses involve proving theorems. Learning how to construct a proof is an essential skill in advanced mathematics.

Practice Problems: Do some practice problems. Understanding the theory is important, but being able to apply it is crucial.

Class Engagement:

Active Participation: Attend all lectures and participate in class discussions. Being actively involved in the class can provide deeper insight into the topics covered.

Office Hours: Utilize your professor's and teaching assistants' office hours to clarify doubts and seek guidance.

Notes: Take diligent notes during lectures, but also consider supplementing them with notes from a textbook or other resources.

- (blank PPT will be shared before class in wechat group)

General Goal

Classic linear algebra training:

How to solve linear equations as fast as possible as humans?

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Computers will do the job!

(Unfortunately, in your exams, you sometime need to solve

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Learn the key ideas and the intuition/reasons behind!

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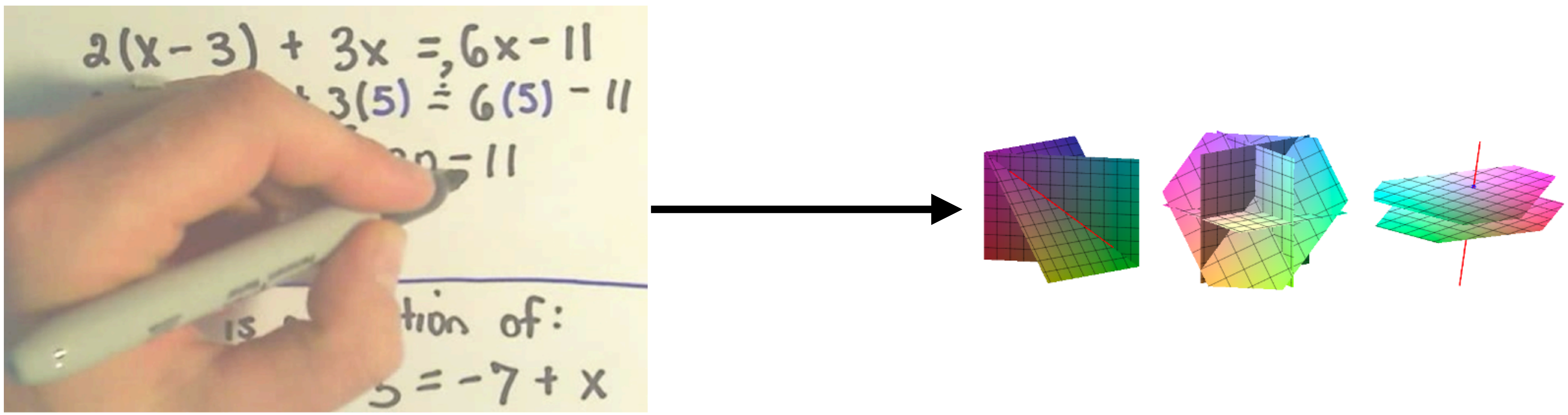

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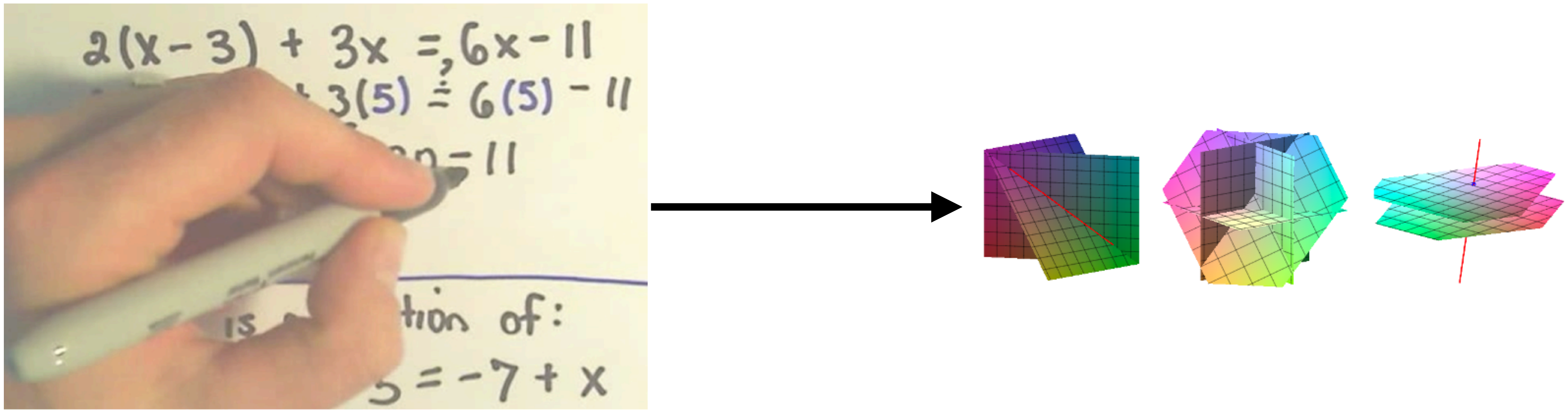


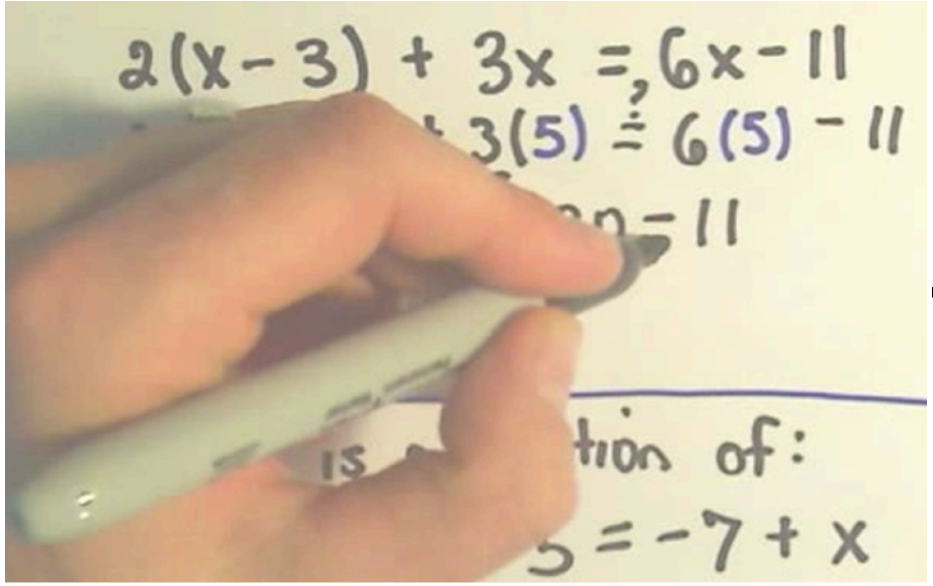
Learn the key ideas and the intuition/reasons behind!

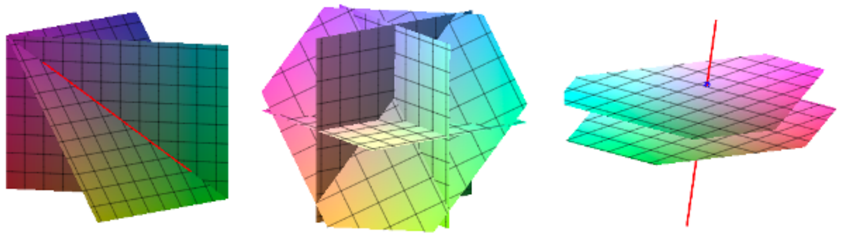
From **How** to **Why (**keep asking yourself why**)**

Today:
Why do we learn data science at all?
Why do we learn linear algebra?
Why do we learn matrix?

Future questions to think:
Why do people define vector products in two ways?
Why do people use Gaussian elimination?
Why are eigenvalues so important?

General Goal





Learn the key ideas and the intuition/reasons behind!

Let's get started!

Part II Vector and Linear Combination

Basic Components in Linear Algebra

Question1: What can you think about the basic components in Linear Algebra from your high school knowledge?

Vectors, matrices, and their operations etc

Question2: What are the most fundamental component in Linear Algebra?

Vectors!

Examples of Vectors

"The world is continuous, but the mind is discrete"

- David Mumford

How to interpret?

Vectors

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]$$

(row) vector

entry/element

Vectors

$$\mathbf{v} = \left[v_1, v_2, \ldots, v_i, \ldots, v_n\right]^\top$$

Transpose

column vector

entry/element

$$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Vectors

Definition (column vectors)

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^\top$$

$\top$ → Transpose

$\mathbf{v}$ → column vector

v_i → entry/element

$$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \equiv (v_1, \ldots, v_n)$$

also written as

Vectors

It is a convention that we consider vectors as "columns"

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^\top$$

$\top$ → Transpose

$\mathbf{v}$ → column vector

v_i → entry/element

$$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \equiv (v_1, \ldots, v_n)$$

$(v_1, \ldots, v_n)$: also written as

Vectors

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^{\top}$$

$\top$ → Transpose

$\mathbf{v}$ → column vector

v_i → entry/element

Example:

A **list** data structure in computer algorithms

A time series (a sequence of data points) in data science problems

Vector Operations

You can't add apples and oranges. This is partially why we need vectors.

Vector Addition

$$\mathbf{v} = (v_1, \ldots, v_n)$$

$$\mathbf{w} = (w_1, \ldots, w_n)$$

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, \ldots, v_n + w_n)$$

Vector Operations

Vector Addition

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$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, \ldots, v_n + w_n)$$

Vector Multiplication

$$\mathbf{v} = (v_1, \ldots, v_n)$$

$$c\mathbf{v} = (cv_1, \ldots, cv_n)$$

Remark: *We do not worry about the domain of the elements* $v_1, \ldots, v_n$ *and* c *so far. They can be chosen from the set of real/complex numbers*

Vector Operations

Vector Addition

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Vector Multiplication

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Element-wise Operations!

Visualization of Vector Operations

Addition $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Visualization of Vector Operations

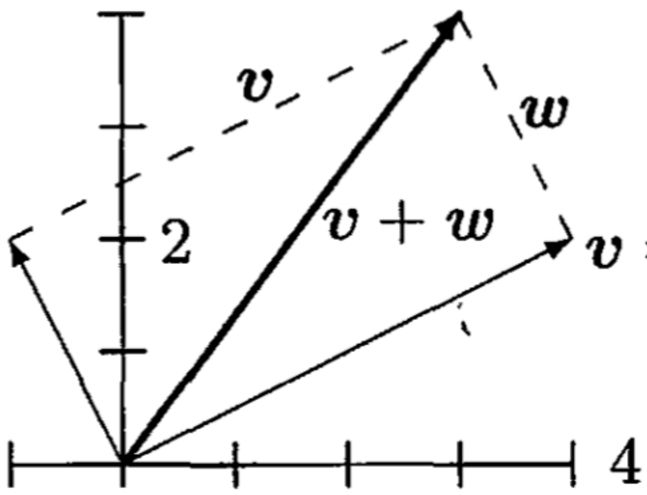
Addition $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Visualization of Vector Operations

Addition $w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

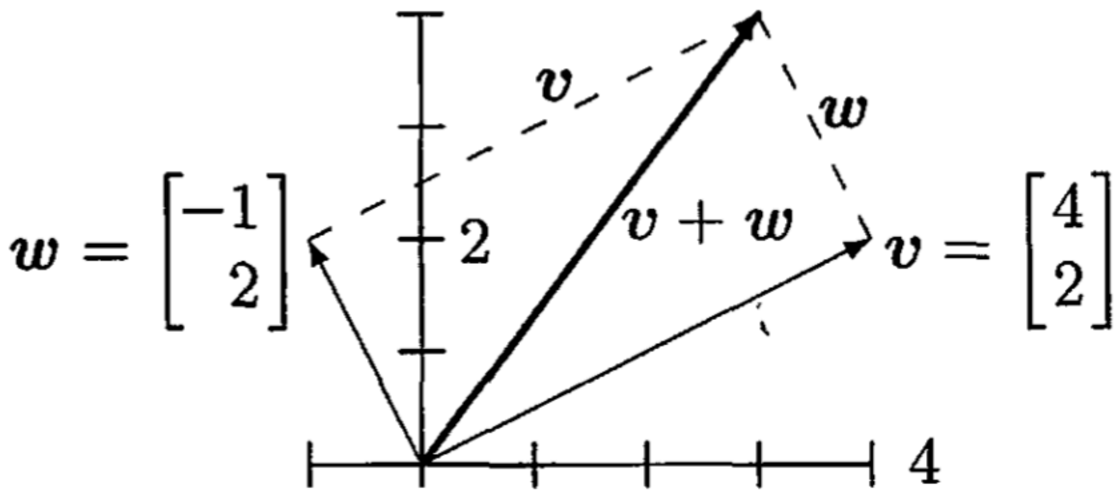
Q: What is the relation between v+w and v, w?

A: ? (Ask students to propose)

Visualization of Vector Operations

Addition $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

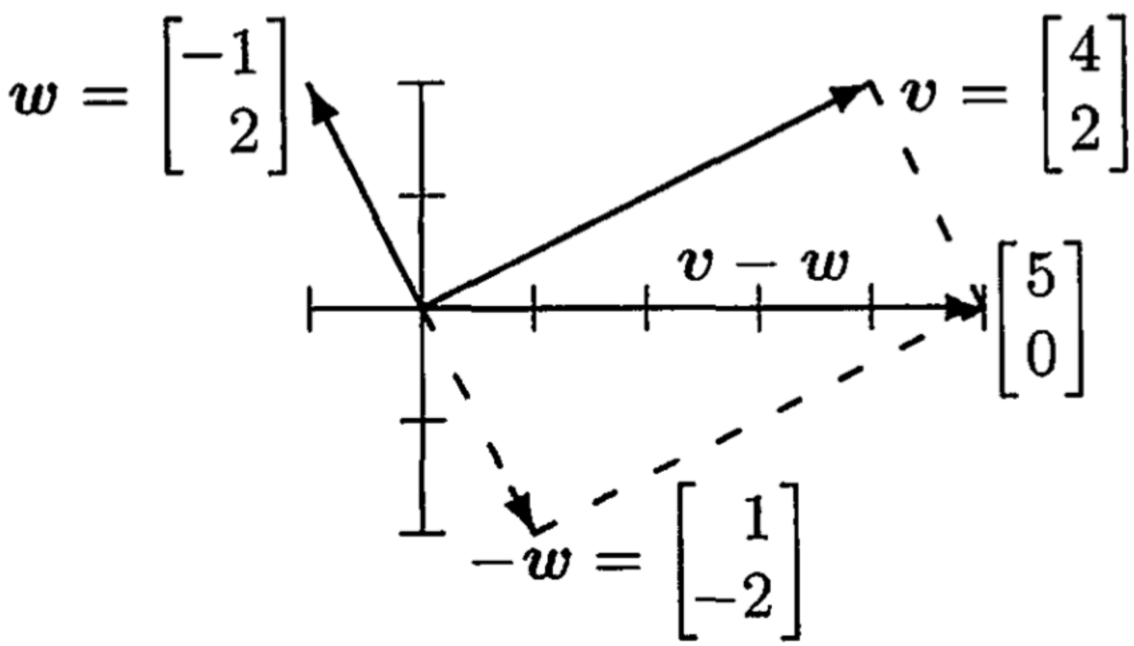


Q: What is the relation between v+w and v, w?

A: Parallelogram.
Form a parallelogram by v and w; v+w is the diagonal line of it.

Visualization of Vector Operations

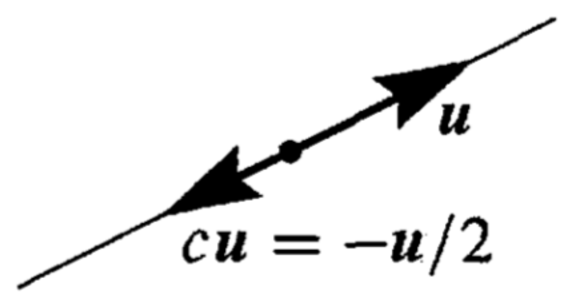
Addition



$$v - w = \begin{bmatrix}4\\2\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix}$$

Visualization of Vector Operations

Multiplication



$$c = -1/2$$

Vector Operations

"**addition**" and "**multiplication**" together form "**linear combination**"

Linear Combination

$$\mathbf{v} = (v_1, \ldots, v_n)$$
$$\mathbf{w} = (w_1, \ldots, w_n)$$
$$c\mathbf{v} + d\mathbf{w} = (cv_1 + dw_1, \ldots, cv_n + dw_n)$$

Definition (Linear Combination of two vectors):

If a vector **u** can be written as c **v** + d **w**, for scalars c, d, then we say **u** is a linear combination of **v** and **w**.

If such scalars c,d do not exist, then u is not a linear Combination of **v** and **w**.

Here, a "scalar" is a real number.

More Examples

Example (Linear Combination of Vectors)

What combination $c\mathbf{v} + d\mathbf{w}$ produces $\mathbf{u}$?

$$\mathbf{v} = (1,2)$$

$$\mathbf{w} = (3,1)$$

$$\mathbf{w} = (14,8)$$

More Examples

Example (Linear Combination of Vectors)

What combination $c\mathbf{v} + d\mathbf{w}$ produces $\mathbf{u}$?

$$c = 2$$

$$d = 4$$

More Examples

Example (Vector Operations)

Find $\mathbf{v}$ and $\mathbf{u}$ such that

$$\mathbf{u} + \mathbf{v} = (4,5,6)$$

$$\mathbf{u} - \mathbf{v} = (2,5,8)$$

More Examples

Example (Vector Operations)

Find $\mathbf{v}$ and $\mathbf{u}$ such that

$$\mathbf{u} = (3,5,7)$$

$$\mathbf{v} = (1,0,-1)$$

Non-Example of Linear Combination

"**Non-example**": in general, $(v_1 w_1,\ v_2 + w_2)$ is NOT a linear combination of (v_1, v_2) and (w_1, w_2)

"Combination", but NOT (necessarily) linear combination

Q: What does "non-example" of a concept mean?
A: For instance, if someone asks "what is fruit", then you can answer:
—"apple, orange are examples of fruit"
—"chairs, masks, meat are non-examples of fruit".
With many examples and non-examples, you get to understand the boundary of a concept.

Q: Where do we use this non-example?
A: People may use such "nonlinear combination" in more advanced nonlinear models, like deep learning

Summary

In today's lecture, we have covered (Textbook Section 1.1)

- Why data science is important;

  Why linear algebra is important for data science

- Definition of **vectors**
- Vector Operations

(Slides will be shared in wechat group, and uploaded to our course webpage some time after class)

Vector Operations

Question: Can you think about any other vector operations?

Vector Operations

Question: Can you think about any other vector operations?

The next lecture!

A General Question: What if we have three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$

Can we write $\mathbf{x}$ as a linear combination of $\mathbf{y}, \mathbf{z}$?

The next next lecture!