Lecture 10

Solving Square Linear System V: Existence and Expression of Inverse

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Today’s Lecture: Key Questions

Consider a square linear system $A\mathbf{x} = \mathbf{b}$

Theorem If A is invertible, then the linear system has a unique solution $x = A^{-1}b$.

Question 1: When is A invertible?

Question 2: How to Express/Compute A^{-1}?

(if exists)

Will provide an answer today.

Today's Lecture: Outline

Today … Existence and expression/computation of inverse.

1. Existence of Inverse
2. Expressions and Computation of inverse

Strang's book: Sec 2.5, 2.6

After this lecture, you should be able to

1. Tell when a matrix is invertible based on pivots
2. Express the inverse of an invertible matrix with the aid of elimination
3. Compute the inverse of a small matrix

Remark & Reminder

Analysis & Algebra

A student commented:

Calculus is more about proofs;

Linear algebra seems to have few proofs, but more concepts.

Well…

Today, we will see some intense proofs.

If LA has 100 levels of difficulty; LA intro course only shows level 1-10.

Today? Around level 3.

PKU LA course: level 10

& I've seen: level 20.

Part 0 Review

5 mins

Effect of Swapping Columns

Swapping columns —-> swapping entries of solution

Claim: Consider a linear system $Ax = b$.
Suppose $\hat{A}$ is obtained by **exchanging column j and column k of A**.
Suppose x^* is a solution of $Ax = b$.
y^* is obtained by **exchanging the jth and kth entry** of x^*.
Then y^* is a solution of $Ay = b$.

Eg 1: Swapping C1 & C2 of A.

(2,1,3)

solution $x = (1,2,3)$ —> solution $y = (1,3,2)$

Eg 2: Swapping C3 & C5 of A.
solution $x = (3,8,7,1,2)$ —> solution $y = (3,8,2,1,7)$

Claim: Only Two Cases of Final Form

Claim If we solve an $n \times n$ square system by Gauss-Jordan Elimination with column exchange, then at the end we obtain one of the two forms:

Form 1: I_n

Form 2: $\begin{bmatrix} I_k & F \\ 0 & B \end{bmatrix}$

Here B is a $(n-k) \times (n-k)$ matrix,

F is a $k \times (n-k)$ matrix.

Remark: If we can obtain Form 1, then column exchange is not needed.

What You (Should) Know by Now

First, how to calculate number of solutions for ANY square linear system.

Highlight: no missing case!

Based on a proof!

Final form

$$[I \mid c] \qquad \left[\begin{array}{cc|c} I & F & c_{1:k} \\ 0 & 0 & c_{k+1:n} \end{array}\right]$$

Second, how to **solve** ANY square linear system

$$\begin{cases} x_1 = \square x_{k+1} + \cdots + \square x_n \\ \vdots \\ x_k = \square x_{k+1} + \cdots + x_n \square \end{cases}$$

(Suppose you know how to swap entries of the final solutions to account for column exchange)

Remark: For writing solution set, we'll study more in Lec 11&12.

Terminology: Gauss Elimination and Gauss-Jordan Elimination

So far, we have been using Gauss Elimination to describe the whole process of solving linear system, for simplicity.

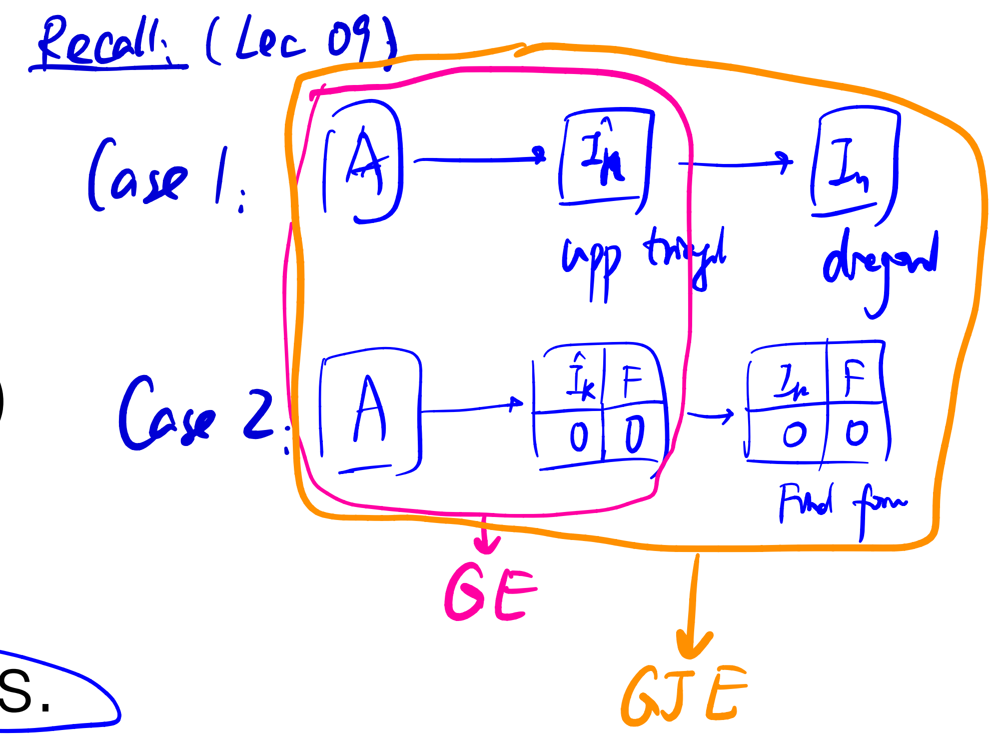
More rigorous terms:

all are (变体) variants of GE

Gauss Elimination:

stop at upper triangular matrix.

(then use back substitution on equations)



Gauss-Jordan Elimination:

continue to eliminate entries above pivots.

Gauss-Jordan Elimination with column exchange:

Use column exchange to ensure diagonal entries to have 1's.

Remark: **Without column exchange, GJE can solve the linear system too. (Discuss in later lectures)**

Part I Inverse of A: Existence

Sec 2.5, Section “Singular versus Invertible”

Outline of This Part

Outline of this part:

Test of invertibility:

i) by pivots;

ii) by equation $Ax=0$

When is A Invertible?

Question 1: When is A invertible?

Diagonal: $A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$, $A^{-1} \exists$ iff $a_{ii} \neq 0, \forall i$.

Upp: $A = \begin{bmatrix} a_{11} & * & \cdots & * \\ & \ddots & & \vdots \\ & 0 & \ddots & * \\ & & & a_{nn} \end{bmatrix}$, $A^{-1} \exists$ iff $a_{ii} \neq 0, \forall i$.

General: $A = \begin{bmatrix} a_{11} & \times & \cdots & \times \\ \times & a_{22} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \times & \times & \cdots & a_{nn} \end{bmatrix}$, $A^{-1} \exists$ iff sth else.

~~$a_{ii} \neq 0, \forall i$.~~

~~too good to be true~~

Exaple: A^{-1} not exist;

but $a_{ii} \neq 0$, $\forall i$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

Prove it later.

When is A Invertible?

Question 1: When is A invertible?

nonzero diagonal entries in final form

Claim 1: If n pivots exist in Gauss-Jordan elimination, then A is invertible.

If P, then Q

If Q, then P.

Claim 2: If A is invertible, then there are n pivots in Gauss-Jordan elimination (with column exchange).

当且仅当 (or sufficient & necessary condn)

Answer 1: A is invertible iff
A has n pivots (assuming A is n by n matrix).

See also Sec 2.5 of Strang's book;
2nd bullet in the beginning of Sec 2.5.

$P \Leftrightarrow Q$. i.e. P iff Q.

Left Inverse and Right Inverse

Definition (left inverse)

If $BA = I$, then B is called the left inverse of A.
left of A

Recall Def of inverse.
If $AB = BA = I$, then $A^{-1} \exists$ and $A^{-1} = B$

Definition (right inverse)

If $AB = I$, then B - - - right inverse of A.

Def. of inverse If B is left inverse of A and B is right inverse of A, then B is the inverse of A

Proof of Claim 1: Step 1
A as Product of Row Elementary Matrices

If there are n pivots

Gaussian-Jordan elimination (GJE) (both forward and backward)

$$A \to A_1 \to A_2 \ldots \to U \to \ldots B_1 \to \ldots\ldots \to I_n$$

$$U = \begin{bmatrix} 1 & * & \ldots & * \\ 0 & 1 & \ldots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \ldots & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 1 \end{bmatrix} = I_n .$$

Corresponding elementary matrices: $E_p, E_{p-1}, \ldots, E_1$.

Then the matrix representation of the whole GE process is

$$E_p E_{p-1} \ldots E_1 A = I_n .$$

$$\overset{?}{\Longrightarrow} \quad A^{-1} = E_p \ldots E_2 E_1 .$$

[just one-side equation;
need extra argument.
Next slide]

Claim 1 If n pivots, then $A^{-1}\ \exists$.

Proof sketch. Step 1 Use GJ-E to obtain $E_p \cdots E_1 A = I$. (1)

$$A \xrightarrow{E_1} A_1 \xrightarrow{E_2} A_2 \longrightarrow \cdots \xrightarrow{E_p} I$$

elementary matrix

(n pivots, get I)

Matrix form:

Answer 1. $\cancel{A E_1 E_2 \cdots E_p = I}$.

Answer 2 $\underbrace{E_p E_{p-1} \cdots E_1}_{M} A = I$. (1)

Step 2. $MA = I \overset{?}{\Longrightarrow} A^{-1} = M$?

where $M = E_p \cdots E_1$.

Proof sketch: n pivots $\overset{\text{get}}{\Longrightarrow} \underbrace{E_p \cdots E_1}_{M} A = I \overset{?}{\Rightarrow} \boxed{A^{-1} \exists}$.

Step 1 Step 2.

Conjecture 1 $MA = I \overset{?}{\Rightarrow} A^{-1} = M$

This is enough to prove Claim 1

BUT

we do NOT prove it today

[Correct]

Incorrect proof: $MA = I$

① $\Rightarrow M\underbrace{AA^{-1}}_{I} = I \cdot A^{-1} = A^{-1}$

② $\Rightarrow M = A^{-1}$

Correct v.s. Proved

teaching & learn: step by step.

avoid circular reasoning → If sth is NOT proved will not use it.

Reminder: For each step of the proof,

think: What result/definition supports it?

Does it appear in the lectures?

Example

$$\left\| \frac{u}{\|u\|} \right\| = 1.$$

$$\overset{?}{=} \frac{\|u\|}{\|u\|} = 1.$$

It's NOT just about "correct",

but emphasize "use result appeared before".

Eg. Some books have a theorem "If $MA = I$, then $M = A^{-1}$", (so this claim is correct)

but you can NOT use it here.

If using theorems in other books is allowed,

then every result in this course can be proved in one sentence.

"Proof: This is a direct corollary of Theorem xx in xx's book."

Conjecture If $MA = I$, then $A^{-1}\ \exists$. [correct, but we don't prove it here. It's more complicated]

$E_p E_{p-1} \cdots E_1 A = I$. $M = E_p \cdots E_1$.

E_i's invertible & product of invertible matrices $E_p \cdots E_1$ is invertible

$(E_p \cdots E_1)^{-1} = E_1^{-1} \cdots E_p^{-1}$.

so M^{-1} exists.

$$\boxed{\text{Claim: If } MA = I, \text{ and } M^{-1}\ \exists, \text{ then } A^{-1}\ \exists.}$$

Proof of Lemma 1

Lemma 1. If $MA = I$, and $M^{-1}\exists$,
then $A^{-1}\exists$.

Proof: Step 1 Definition of inverse.

Want to prove: $A^{-1}\exists$, i.e.

$MA = I$ and $AM = I$. (*)

Analysis. Just write down definition and conditions. Even high school students can do these 2 steps.

Step 2 Conditions:

$MA = I$ (1)

$M^{-1}\exists$ (2)

Step 3 Play with conditions

Multiply ~~A, AM,~~ M^{-1}, $\quad M^{-1}(MA) = M^{-1}\cdot I$

$\Rightarrow \quad A = M^{-1}$ (3)

Step 4 $A = M^{-1} \Rightarrow A\cdot M = (M^{-1})\cdot M = I$. (*) is proved.

Proof of Claim 1: Step 2
Left Inverse is Invertible ===> Itself Invertible

$$\underbrace{E_pE_{p-1}\ldots E_1}_{M} A = I_n \overset{?}{\Longrightarrow} A^{-1} = E_p\ldots E_2E_1\,.$$

i.e. $$MA = I_n \overset{?}{\Longrightarrow} A^{-1} = M\,.$$

Extra property: $M = E_p \text{--} E_1$ is invertible.

Just left inverse. Need to show $AM = I_n$? Seems nontrivial?

Lemma 1: If $MA = I_n$, and M is invertible, then $A^{-1} = M$.

Proof: $M^{-1}\ \exists$ so

$$MA = I \Rightarrow M^{-1}(MA) = M^{-1}I \Rightarrow A = M^{-1} \quad (1)$$

Then $$A \cdot M \overset{(1)}{=} M^{-1}M = I.$$

$$MA = I \quad \text{(Condition)}$$

$$\left.\begin{matrix} A \cdot M = I \\ MA = I \end{matrix}\right\} \underset{\text{def of inverse}}{\Longrightarrow} A^{-1}\ \exists \ \& \ A^{-1} = M.$$

Roadmap of Proof of Claim 1.

Summarize the proof.

n pivots $\Longrightarrow$ $A^{-1}\ \exists$

$E_p \cdots E_1 A = I$

$(E_p \cdots E_1)^{-1}\ \exists$ by Lemma 1

[① elementary matrix E_j's are invertible
② product of invertible matrix invertible]

Proof of Claim 2: Skipped

Claim 2: If A is invertible, then there are n pivots in Gauss-Jordan elimination.

$P \Rightarrow Q$

Contrapositive

$\bar{Q} \Rightarrow \bar{P}$.

Proof: Skipped. (not prove it).
(See Strang's book 5th edition page 88, after "Reasoning in reverse")

Need to prove by contradiction (反证法).

To prove. (Contrapositive 逆否)

If $< n$ pivots. then A is NOT invertible.

Case 2

$A \xrightarrow{\text{row, col.}} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ not invertible

Need to show: If $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ is not invertible,

then A is not invertible. [need extra steps]

When is A Invertible?

$P \Rightarrow Q$

contraposi. not $Q \Rightarrow$ not P

Question 1: When is A invertible?

P $\Longleftrightarrow$

Answer 2: A is invertible iff
$Ax = 0$ has a unique solution $x = 0$.

Q

See also Sec 2.5 of Strang's book;
4th bullet in the beginning of Sec 2.5.

$P \Leftarrow Q$

Proof:

"$\Longleftarrow$" "If": If $Ax=0$ has unique solution, then A is invertible. (S1)

$\Longleftrightarrow$ (contrapositive) If $A^{-1} \not\exists$, then $Ax=0$ has 0 or >1 solutions.

逆否

$P \Rightarrow Q$

"$\Rightarrow$" "Only if": If $A^{-1} \exists$, then $Ax=b$ has unique solution, (S2)

Why? $A^{-1}(Ax) = A^{-1}b \Rightarrow x = A^{-1}b$; & $A \cdot (A^{-1}b) = b$,

$\Downarrow$

x is unique solution.

You've seen one of them

Roadmap of Proof of Claim 2.

$$\boxed{A^{-1} \nexists} \Longrightarrow \boxed{< n \text{ pivots}} \Longrightarrow \boxed{Ax=b \text{ has } 0 \text{ or } >1 \text{ solutions.}}$$

Lec 9. (Case 2: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$)

Remark. For square matrix A,

A is invertible

$\iff Ax=0$ has unique solution

$\iff Ax=b$ has unique solution for any b.

Recall: Two Cases for Square Systems

End of backward step of GJE

Coefficient matrix $\quad$ b

Case 1: n pivots

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} c \quad \text{correspond to} \quad \begin{cases} x_1 = 0, \\ x_2 = 0,, \\ \vdots \\ x_n = 0,, \end{cases}$$

Case 2: < n pivots. (Allow column Exchange)

$$\begin{bmatrix} 1 & 0 & \dots & 0 & * & * & * \\ 0 & 1 & \dots & 0 & * & * & * \\ \vdots & \vdots & \ddots & \ddots & * & * & * \\ 0 & 0 & \dots & 1 & * & * & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix} \quad \text{correspond to} \quad \begin{cases} x_{i_1} = \dots, \\ \vdots \\ x_{i_k} = \dots, \\ 0 = 0,, \\ \vdots \\ 0 = 0. \end{cases}$$

Lec 9

If $c_i = 0, \forall i$, ∞

If $c_i \neq 0$, no soln

Remark: GE in the textbook does NOT perform column exchange.
Anyhow, here our goal is to study # of solutions, so column exchange is OK.

Claim: Consider Ax = 0. If less than n pivots, then Ax =0 has ∞-many solution.

Remark: For general b, if < n pivots, then Ax=b has no or ∞-many solutions

(Summary of Lec 09)

Part II Computing Inverse

Sec 2.5, Section “Calculating A^{-1} by …”

Outline of This Part

Outline of this Part:

—Inverse computation I: Naive way.

—Inverse computation II:
Smarter way by row operations.

Expression of Inverse

Question 2a: How to Express A^{-1}? (if exists)

Answer:
Suppose $E_1, \ldots, E_k, E_{k+1}, E_{k+2}, \ldots, E_p$ are the elementary matrices corresponding to the operations in GJE to get an identity matrix.
(if A invertible, then we indeed get identity matrix by GJE)

Then

$M =$

$$A^{-1} = E_p \ldots E_2 E_1 . \qquad (10.1)$$

Claim: A matrix is invertible iff
it can be written as the product of elementary matrices.

Recall: Quadratic Equations [Reading material]

During middle school, when you learn quadratic equations, what do you learn?

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

To solve $x^2 + 4x + 1 = 0$,
write it as $(x+2)^2 = 3$,
Then get $x + 2 = \sqrt{3}$ or $-\sqrt{3}$.

I know you know these. But…
What are them?
I mean…. WHAT are them?

Express v.s. Compute [Reading material]

Express: a clear formula containing specific symbols with clear meanings.

Eg1 $A^{-1} = U^{-1}L^{-1}P$.

Eg2 $A^{-1} = E_p \ldots E_2 E_1$

Eg3 The solution of $Ax = b$ is $x = A^{-1}b$ when A is invertible

Eg4 One root of $ax^2 + bx + c = 0$ is $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Express v.s. Compute [Reading material]

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Eg4 One root of $ax^2 + bx + c = 0$ is $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Compute: a procedure (algorithm) to obtain desired answers for any concrete numbers.

Eg1 $A^{-1} = E_p \ldots E_2 E_1$ can compute the inverse (next pages)

Eg2 GE can compute the solution

(no formula of the root needed)

Eg3 completing square (配方法) can compute roots of $ax^2 + bx + c = 0.$

(no formula of the root)

Express v.s. Compute [Reading material]

Express: a clear formula containing specific symbols with clear meanings.

Eg1 $A^{-1} = U^{-1}L^{-1}P$.

Eg2 $A^{-1} = E_p \ldots E_2 E_1$

Eg3 The solution of $Ax = b$ is $x = A^{-1}b$ when A is invertible

Eg4 One root of $ax^2 + bx + c = 0$ is $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

Compute: a procedure (algorithm) to obtain desired answers for any concrete numbers.

Eg1 $A^{-1} = E_p \ldots E_2 E_1$ can compute the inverse (next pages)

Eg2 GE can compute the solution of linear system

(no formula of the solution needed)

Eg3 completing square (配方法) can compute roots of $ax^2 + bx + c = 0.$

(no formula of the root)

Relation:

1) Expression can be used to compute, if each symbol can be computed.

2) But expressions do not have to be computable

e.g., if it contains symbols that are not easy to compute (e.g. U^{-1})

3) Algorithms can help derive expressions sometimes, e.g. GE —> A^{-1}

Computing Inverse

Question 2b: How to Compute A^{-1}? (if exists)

Algorithm 1 (compute A^{-1})

Step 1: **Forward elimination**.

Run forward elimination, till get upper triangular matrix U.
IF U contains zero diagonal entry:
STOP and report: No inverse.
ELSE Go to Step 2.

Step 2: **Backward substitution.**

Run backward substitution, till get identity matrix I_n.

Computing Inverse

Question 2b: How to Compute A^{-1}? (if exists)

Algorithm 1 (compute A^{-1})

Step 1: **Forward elimination**.

Run forward elimination, till get upper triangular matrix U.

IF U contains zero diagonal entry:

STOP and report: No inverse.

ELSE Go to Step 2.

Step 2: **Backward substitution.**

Run backward substitution, till get identity matrix I_n.

Step 3 Record elementary matrices.

Record elementary matrices $E_1, \ldots, E_k$ in Step 1.

Record elementary matrices $E_{k+1}, E_{k+2}, \ldots, E_p$ in Step 2.

Step 4: **Compute inverse**.

Compute $$A^{-1} = E_p \ldots E_2 E_1 . \qquad (10.2)$$

2 by 2 Example: Understandable Process

Step 1 & 2: GE.

$$\begin{bmatrix}2 & 1\\6 & 8\end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix}2 & 1\\0 & 5\end{bmatrix} \xrightarrow{-\frac{1}{5}R_2+R_1} \begin{bmatrix}2 & 0\\0 & 5\end{bmatrix} \xrightarrow{\substack{\frac{1}{2}R_1\\ \frac{1}{5}R_2}} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Step 3: Express as matrix multiplication (1st and 2nd step):

$$A = \begin{bmatrix}2 & 1\\6 & 8\end{bmatrix}$$

$$E_{-3R_1+R_2}A = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\6 & 8\end{bmatrix} = \begin{bmatrix}2 & 1\\0 & \end{bmatrix} \triangleq U\,.$$

$$E_{-\frac{1}{5}R_2+R_1}U = \begin{bmatrix}1 & -1/5\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\0 & 5\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 5\end{bmatrix} = D\,.$$

$$E_{\frac{1}{5}R_2}E_{\frac{1}{2}R_1}D = I_2\,.$$

2 by 2 Example: Understandable Process

Step 1 & 2: GE.

$$\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2 + R_1} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 3: Express as matrix multiplication (1st and 2nd step):

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$$

$$E_{-3R_1+R_2}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & \end{bmatrix} \triangleq U\,.$$

$$E_{-\frac{1}{5}R_2+R_1}U = \begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D\,.$$

$$E_{\frac{1}{5}R_2}E_{\frac{1}{2}R_1}D = I_2\,.$$

Step 4: Compute inverse by the elementary matrices

Thus $E_{\frac{1}{5}R_2}E_{\frac{1}{2}R_1}E_{-\frac{1}{5}R_2+R_1}E_{-3R_1+R_2}A = I_2\,.$

Thus $A^{-1} = E_{\frac{1}{5}R_2}E_{\frac{1}{2}R_1}E_{-\frac{1}{5}R_2+R_1}E_{-3R_1+R_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

2 by 2 Example: Simplified Process, i.e., Algorithm 1

Step 1 & 2: GE.

$$\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2 + R_1} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 3 Record elementary matrices in Step 1&2

2 by 2 Example: Simplified Process, i.e., Algorithm 1

Step 1 & 2: GE.

$$\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2 + R_1} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 3 Record elementary matrices in Step 1&2

$$E_1 = E_{-3R_1+R_2} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad E_2 = E_{-\frac{1}{5}R_2+R_1} = \begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix}.$$

$$E_3 = E_{\frac{1}{2}R_1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_4 = E_{\frac{1}{5}R_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}.$$

Step 4: Write the inverse by formula (9.1).

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

2 by 2 Example: Algorithm 1 with Multiplication Trick

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

You can use definition to perform multiplication.

But these are elementary matrices; faster way? Row operation!

$$\begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \Longleftrightarrow$$ Applying $-\frac{1}{5}R_2 + R_1$ to the matrix $A_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$,

So we get $A_2 = \begin{bmatrix} 8/5 & -1/5 \\ -3 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2+R_1} \begin{bmatrix} \frac{8}{5} & -\frac{1}{5} \\ -3 & 1 \end{bmatrix}$$

2 by 2 Example: Algorithm 1 with Multiplication Trick

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

You can use definition to perform multiplication.

But these are elementary matrices; faster way? Row operation!

$\begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \iff$ Applying $-\frac{1}{5}R_2 + R_1$ to the matrix $A_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$,

So we get $A_2 = \begin{bmatrix} 8/5 & -1/5 \\ -3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 8/5 & -1/5 \\ -3 & 1 \end{bmatrix} \iff$ Applying $\frac{1}{5}R_2$ to the matrix $\begin{bmatrix} 8/5 & -1/5 \\ -3 & 1 \end{bmatrix}$, so we get $A_3 = \begin{bmatrix} 8/5 & -1/5 \\ -3/5 & 1/5 \end{bmatrix}$

$\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8/5 & -1/5 \\ -3/5 & 1/5 \end{bmatrix} \iff$ Applying $\frac{1}{2}R_1$ to the matrix $\begin{bmatrix} 8/5 & -1/5 \\ -3/5 & 1/5 \end{bmatrix}$, so we get $A_4 = \begin{bmatrix} 8/10 & -1/10 \\ -3/5 & 1/5 \end{bmatrix}$

Observation: Same sequence of operations as GE.

Algorithm 2: Applying Same Operations to I

Module 1: GE.

$$\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2+R_1} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A E1 A1 In

Module 2: Apply same operation to I.

$$A^{-1} = E_{\frac{1}{5}R_2} E_{\frac{1}{2}R_1} E_{-\frac{1}{5}R_2+R_1} E_{-3R_1+R_2} I_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2+R_1} \begin{bmatrix} 8/5 & -1/5 \\ 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 8/10 & -1/10 \\ -3/5 & 1/5 \end{bmatrix}$$

I E1 B1 A^{-1}

Last page:

$$\begin{tikzcd}
A \arrow[r, "E_1"] & A_1 \arrow[r, "E_2"] & \cdots \arrow[r, "E_k"] & A_k = I \\
I_n \arrow[r, "E_1"] \arrow[u, phantom, "I_n"] & B_1 \arrow[r, "E_2"] & B_2 \cdots \arrow[r, "E_k"] & B_k = A^{-1}
\end{tikzcd}$$

$A^{-1} = E_k E_{k-1} \cdots E_1$

General fact: $P \xrightarrow{E_1} P_1 \rightarrow \cdots \xrightarrow{E_k} P_k = Q$,

$\iff (E_k E_{k-1} \cdots E_1) P = Q$

$\iff A^{-1} P = Q$

Special case 1 Let $P = A$. Then $A^{-1} \cdot A = I$. (GJE)

$A \xrightarrow{E_1, \cdots, E_k} I$.

Special case 2 Let $P = I$. Then $A^{-1} \cdot I = A^{-1}$.

$I \xrightarrow{E_1 \cdots E_k} A^{-1}$.

$[A, I] \xrightarrow{A^{-1}} [I, A^{-1}]$

$$E_p E_{p-1} \cdots E_1$$

$$[A, I] \xrightarrow{A^{-1}} [I, A^{-1}]$$

GJE

computing A^{-1}

same seq. of operation

different starting point A or I or ...

Decouple initial matrix P, and operation $E_p \cdots E_1$

Algorithm 2

Module 1: GE. $A \xrightarrow{\textbf{op 1}} \square \xrightarrow{\textbf{op 2}} \square \ldots \xrightarrow{\textbf{op k}} I_n$.

Module 2: Apply same operation to I.

$$I_n \xrightarrow{\textbf{op 1}} \square \xrightarrow{\textbf{op 2}} \square \ldots \xrightarrow{\textbf{op k}} A^{-1} .$$

Algorithm 2

Module 1: GE. $A \xrightarrow{\text{op 1}} \square \xrightarrow{\text{op 2}} \square \dots \xrightarrow{\text{op k}} I_n$.

Module 2: Apply same operation to I.

$I_n \xrightarrow{\text{op 1}} \square \xrightarrow{\text{op 2}} \square \dots \xrightarrow{\text{op k}} A^{-1}$.

GJE

Algorithm 2: Apply ~~GE~~ to $[A, I_n]$

$[A, I_n] \xrightarrow{\text{op 1}} \square \xrightarrow{\text{op 2}} \square \dots \xrightarrow{\text{op k}} [I_n, A^{-1}]$.

only work for the case that A^{-1} exists.
(next page)

Justification: GE is essentially multiplying A^{-1},
Applying to A leads to I_n.
Thus Applying to I_n leads to A^{-1} .

<u>Remark</u>. No column exchange is allowed when computing A^{-1}.

Column exchange is NOT needed:

If you see

$$\begin{bmatrix} 1 & \times & \times & - & - & - \\ 0 & 1 & \times & - & - & - \\ 0 & 0 & 0 & \times & \cdot & \times \\ 0 & 0 & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \times & & \times \end{bmatrix} \quad \text{i.e.} \quad \left[\begin{array}{c|c|c} \hat{I}_k & \begin{matrix}\times\\ \vdots\\ \times\end{matrix} & H \\ \hline 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & J \end{array}\right]$$

(i.e. column exchange is needed),

then you won't get n pivots in the end,

which implies A^{-1} does NOT exist.

Since column exchange is not needed for computing A^{-1},

we use "GJE", NOT "GJE with column exchange".

Algorithm 2: a 3 by 3 Example

Problem: Find the inverse of $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

Solution:

$$[A|\mathrm{I}] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

Thus the inverse of A is $A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$

Q: Compute A^{-1}.

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ \underline{2} & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\to$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

Summary of II. 1 Algorithms for Computing Inverse

1) **Algorithm 1**: compute A^{-1} by $A^{-1} = E_p \dots E_2 E_1$.

Here $E_1, \dots, E_k, E_{k+1}, E_{k+2}, \dots, E_p$ are elementary matrices during GE (to get an identity matrix)

Bottom line:

Do you know how to get $E_i, \forall i$, and multiply matrices?

If so, then you know how to compute A^{-1}

Summary of II. 1 Algorithms for Computing Inverse

1) **Algorithm 1**: compute A^{-1} by $A^{-1} = E_p \dots E_2 E_1$.

Here $E_1, \dots, E_k, E_{k+1}, E_{k+2}, \dots, E_p$ are elementary matrices during GE (to get an identity matrix)

matrix rep. of op

Bottom line:

Do you know how to get $E_i, \forall i$, and multiply matrices?

If so, then you know how to compute A^{-1}

Reminder:
Inverse may not exist

2) **Algorithm 2:** compute $A^{-1} = E_p \dots E_2 E_1 I_n$ by applying elementary operations to I_n .

operation

matrix

Bottom line:

Do you know how to conduct GE?

If so, then you know how to compute A^{-1}

Appendix: Another Proof of "Inverse Exists iff n pivots"

[Reading material]

Remark on Difficulties

First, right inverse.

Use GE matrix representation, can only prove:

If there are n pivots, then there exists left inverse of A.

Need to: a) use GJE to [A, I] to show right inverse exists;

b) Then show left inverse = right inverse.

Second, not easy to prove the reverse direction:

If A is invertible, then there must be n pivots.

Method 1 (Textbook): prove by contradiction;

requires 4 steps; requires deeper understanding of GE.

Method 2 (next): use PLU decomposition

When is A Invertible?

Question 1: When is A invertible?

We will utilize the theorem in Lec 9 to answer the question.

$$PA = LU,$$

where P is permutation matrix, L is lower triangular, U is upper triangular.

$A^{-1} \exists$ iff $U^{-1} \exists$ iff $U_{ii} \neq 0 \ \forall i$

i.e. n pivots

Recall: Two Properties Learned Before

Property 1: Product of invertible matrix is invertible.

Property 2: Permutation matrix is invertible.

When is A Invertible? First Lemma

Lemma 1: A is invertible iff U is invertible.

Suppose $PA = LU$.
Fact: P, L are invertible.

Proof: "**If part**". If U is invertible, then

"**Only if part**". If A is invertible, then

When is A Invertible? Second lemma

Lemma 2: U is invertible iff $u_{ii} \neq 0$, $\forall i \in \{1,2,\ldots,n\}$.

Fact: U is an upper triangular matrix.

Thus Lemma 2 holds due to Property 9.2 in earlier slides.

Combine Lemma 1 and Lemma 2,
A is invertible $\Longleftrightarrow$ U is invertible;
$\Longleftrightarrow u_{ii} \neq 0$, $\forall i \in \{1,2,\ldots,n\}$.

When is A Invertible?

Question 1: When is A invertible?

Gaussian elimination (GE) (forward part, allow row exchange)

$$A \to A_1 \to A_2 \ldots \to U$$

U is upper triangular.

Theorem 2: Suppose PA = LU is the decomposition given in Thm 1.
Then A is invertible iff all diagonal entries of U are nonzero;

When is A Invertible?

Question 1: When is A invertible?

Gaussian elimination (GE) (forward part, allow row exchange)

$$A \to A_1 \to A_2 \ldots \to U$$

U is upper triangular.

Theorem 2: Suppose PA = LU is the decomposition given in Thm 1.
Then A is invertible iff all diagonal entries of U are nonzero;

Recall: Non-zero diagonal entries of U are the pivots (of A).

Answer 1: A is invertible iff
A has n pivots (assuming A is n by n matrix).

See also Sec 2.5 of Strang's book;
2nd bullet in the beginning of Sec 2.5.

Summary Today (write Your Own)

One sentence summary:

Detailed summary:

Summary Today (of Instructor)

One sentence summary:

We study the test conditions and computation of inverse.

Detailed summary:

1. **Test conditions**
   —Algorithm test: n pivots
   —Equation test: Ax = b has a unique solution
   —can be written as product of elementary matrices

not easy

(not easy as ~~Ax=b~~)

2. **Expressions and computation of inverse**
   —Expression $A^{-1} = E_p \dots E_2 E_1$. (10.1)
   —Algorithm 1: Use (10.1).
   —Algorithm 2: apply GE to [A, I] to get [I, A^{-1}]

take the

A^{-1}? $Ax = b \iff x = A^{-1}b$

3. Time complexity
   —Vector addition and multiplication: $O(n)$
   —Matrix-vector multiplication: $O(n^2)$
   —Matrix-matrix multiplication: $O(n^3)$
   —Gaussian elimination: $O(n^3)$

Mid-term Nov 5, 16:30 - 18:30.

Hw 3 released soon.

Check-in