# Lecture 11

## *Structure of Solution Set I: RREF and Linear Space*

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# Today's Lecture: Outline

Today … Linear Space and Solution Set

1. Motivation: Solution set of rectangular system

2. Linear space

3. Null space and column space

Strang's book: Sec 3.1, 3.2

# Today's Lecture: Learning Goals

After this lecture, you should be able to

1. Verify linear space

2. Tell a few common linear spaces

3. Explain why the solution set of Ax=0 is a linear space

4. Write the column space of a matrix

# Part 0 Review and Future Roadmap

## Main Theorem of Lec 10

**Theorem 11.1** (Equivalent Conditions for Invertibility)

Let $A \in \mathbb{R}^{n \times n}$.

The following statements are equivalent:

1. $A$ is invertible
2. The linear system $A\mathbf{x} = 0$ has a unique solution $\mathbf{x} = \mathbf{0}$
3. $A$ is a product of elementary matrices
4. After GJE of A, the final form is $I_n$
5. More …

We have proved the equivalence of 1,2,3,4.

# Part I  Solving Rectangular System

# Example of Rectangular System

**Example 1:**

Matrix View

$$\left\{\begin{array}{l} x_1 + x_2 - x_3 = 6 \\ 2x_1 + 4x_2 + 2x_3 = 20. \end{array}\right.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 4 & 2 & 20 \end{array}\right]$$

Equation View

# Relation of Rectangular and Square System

The process seems quite similar to solving a square system (breakdown case)

Any relation between solving these two types of systems?

## Transform Rectangular System to Square System (1): Wide Coefficient Matrix

**Claim**: If you can solve all square systems, then
you can solve all rectangular linear systems.

### Example 1 (revisited):

$$\begin{cases} x_1 + x_2 - x_3 = 6 \\ 2x_1 + 4x_2 + 2x_3 = 20. \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 4 & 2 & 20 \end{array}\right]$$

Trick: Add ______________________________, obtain:

Equation View

$$\begin{cases} x_1 + x_2 - x_3 = 6 \\ 2x_1 + 4x_2 + 2x_3 = 20 \\ 0x_1 + 0x_2 + 0x_3 = 0. \end{cases}$$

Matrix View

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 4 & 2 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

# Transform Rectangular System to Square System (2): Tall Coefficient Matrix

**Claim**: If you can solve all square systems, then you can solve all rectangular linear systems.

**Example 2:**

$$\begin{cases} x_1 + x_2 = 6 \\ 2x_1 + 4x_2 = 20 \\ x_1 + 2x_2 = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 4 & 2 & 20 \end{array}\right]$$

Equation View

$$\begin{cases} x_1 + x_2 - x_3 = 6 \\ 2x_1 + 4x_2 + 2x_3 = 20 \\ 0x_1 + 0x_2 + 0x_3 = 0. \end{cases}$$

Matrix View

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 4 & 2 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

# Example of Rectangular System

**Example 1:** RREF is

$$\begin{cases} x_1 + x_2 = 12 \\ 2x_1 + 4x_2 = 38. \end{cases}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

The equivalent system is

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 0, \\ x_2 + x_3 - 3x_4 &= 0, \\ 0 &= 1. \end{aligned}$$

Solution set:

# Rectangular Case 1: No Solution

**Example 1:** RREF is

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

The equivalent system is

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 0, \\ x_2 + x_3 - 3x_4 &= 0, \\ 0 &= 1. \end{aligned}$$

Solution set:

# Rectangular Case 2: Infinitely many solutions

**Example 2:** RREF is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equivalent linear system:

$$\begin{aligned} x_1 + x_3 &= 3, \\ x_2 - x_3 &= 2. \end{aligned}$$

Equivalent to:

$$\boxed{\begin{aligned} x_1 &= 3 - x_3, \\ x_2 &= 2 + x_3. \end{aligned}}$$

Solution set:

$$S = \left\{ \begin{bmatrix} 3 - t \\ 2 + t \\ t \end{bmatrix} \middle| t \in \mathcal{R} \right\}$$

## Rectangular Case 2: Infinitely many solutions

Example 3

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The final the equivalent linear system is:

$$\begin{aligned} x_1 + x_2 &= 3, \\ x_3 + x_4 &= -4, \\ x_5 &= 2, \\ 0 &= 0. \end{aligned}$$

We see that $x_3 + x_4 = -4$ does not uniquely determine the value of $x_3, x_4$, and $x_1 + x_2 = 3$ does not uniquely determine the value of $x_1, x_2, x_4$.

$$\begin{aligned} x_1 &= 3 - x_2, \\ x_3 &= -4 - x_4, \\ x_5 &= 2 \end{aligned}$$

**Solution set:**

# Rectangular Case 2: Infinitely many solutions

Example 3 $\begin{bmatrix} 1 & 0 & 0 & -1 & 4 & | \ 5 \\ 0 & 1 & 0 & 3 & 1 & | \ 2 \\ 0 & 0 & 1 & 2 & 2 & | \ 1 \\ 0 & 0 & 0 & 0 & 0 & | \ 0 \end{bmatrix}$

**Equivalent linear system:**

$$\begin{cases} x_1 - x_4 + 4x_5 = 5, \\ x_2 + 3x_4 + x_5 = 2, \\ x_3 + 2x_4 + 2x_5 = 1, \\ 0 = 0. \end{cases} \Longrightarrow \begin{cases} x_1 = x_4 - 4x_5 + 5, \\ x_2 = -3x_4 - x_5 + 2, \\ x_3 = -2x_4 - 2x_5 + 1, \\ 0 = 0. \end{cases}$$

**Solution set:**

## What’s Special about Solution Set?

What’s special about the solution set?

Recall: Cannot have 2 solutions; must be 0, 1 or infinity.

**More to say about the solution set?**

## Geometrically: Solution Set is _______

2D:

3D:

# Solution of Systems?

**Known known**
**(After Lec 10):**
**Do I know how to solve all "good" square linear systems?**

Yes.

Theory guarantees.

**Unknown ??**
**(After Lec 10):**
**Do I know how to solve all rectangular linear systems?**

**?**
I don't know whether I know or not.

**Q1**: Have we enumerated all cases?

**Q2:** What is a proper way of expressing the solution set?

Need theory!

# Part II  Linear Space

—Linear space
—Subspace

# Def: Linear Space

## Definition 11.1 (Linear space)

Suppose V is a set associated with two operations:
(i) Addition "+": $\mathbf{u}+\mathbf{v} \in V, \forall \mathbf{u} \in V, \mathbf{v} \in V$.
(ii) Scalar multiplication: $\alpha \mathbf{u} \in V, \forall \alpha \in \mathbb{R}, \mathbf{u} \in V$.

V is called a linear space over $\mathbb{R}$ if the 8

(A1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \forall\ \mathbf{u},\ \mathbf{v} \in V$ .

(A2) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+\mathbf{v}+\mathbf{w}, \forall\ \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$.

(A3) There exists a element $\mathbf{0}$ s.t. $\mathbf{u}+\mathbf{0}=\mathbf{u}$, $\forall\ \mathbf{u} \in V$.

(A4) If $\mathbf{u} \in V$, then there exists $-\mathbf{u}=(-1)\mathbf{u}$, s.t. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(A5) $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}, \forall\ \alpha \in \mathbb{R},\ \mathbf{u},\ \mathbf{v} \in V$.

(A6) $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}, \forall\ \alpha,\ \beta \in \mathbb{R},\ \mathbf{u} \in V$.

(A7) $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}, \forall\ \alpha,\ \beta \in \mathbb{R},\ \mathbf{u} \in V$.

(A8) $1\mathbf{u}=\mathbf{u}$.

First time in your study:
Define a subject by "structure"!

**Remark**: Linear space is also called "vector space" (in textbook as well); we will use linear space more often.

# Linear Space

**Informally:**

**Linear space** is a set:

i) that is equipped with addition and scalar multiplication;

ii) any linear combination of elements is in this space.

| | |
|---|---|
| category | Set + |
| operation | Linear combination + |
| property | Closure under linear combination |

# Example: Euclidean space 欧氏空间

**Eg 1:** $\mathbb{R}^n$ is a linear space, called $n$-dimensional Euclidean space.

**Remark**: Don't take it for granted!
It can be verified by definition that $\mathbb{R}^n$ is a linear space.

**Verification:**

—Vector addition, scalar multiplication satisfy the required axioms of linear space.

—Any linear combination of vectors is also a vector.

—Existence of zero element, inverse of addition, unit element:
$0 + \mathbf{u} = \mathbf{u}$; $\mathbf{u} + (-\mathbf{u}) = 0$; $1\mathbf{u} = \mathbf{u}$.

What operation is NOT needed for verification?

# Example: Euclidean space 欧氏空间

## Geometrical interpretation:

$n = 1: \quad R^1$ is a line.
Each element corresponds to a point on the line.

$n = 2: \quad R^2$ is a plane.
Each element corresponds to a point on the plane.

$n = 3: \quad R^3$ is a 3-dimensional space.
Each element corresponds to a point in the space.

**Remark**: The name "space" comes from the geometrical term "space" (e.g. lines forms plane; planes form space); But can be more than 3D.

# Example: matrix space

**Eg 2:** $\mathbb{R}^{m \times n}$ is a linear space, called a matrix space.

**Remark**: Don't take it for granted!
It can be verified by definition.

**Verification:**

—Matrix addition, scalar multiplication satisfy the required axioms of linear space.

—Any linear combination of matrices is also a matrix of the same dimension.

—Existence of zero element, inverse of addition, unit element:

$$0 + A = A;\ A + (-A) = 0; \quad 1 \cdot A = A\,.$$

What operation is NOT needed for verification?

# Example: polynomial space

**Eg 3:** Set of polynomials with degree no more than $k$ is a linear space.

**Verification:**

# Non-Examples

In the following, assume the set is equipped with regular addition and scalar multiplication of real numbers.

**NonEg 1:** $\{0,1,2,3,4,\ldots\}$ is not a linear space.

Why?

**NonEg 2:** Set of non-negative real numbers $\mathbb{R}_+$ is not a linear space.

Why?

# Exercise

In the following, assume the set is equipped with standard addition and scalar multiplication.

Are the following linear spaces?

1) Set of $n \times n$ upper triangular matrices.

2) Set of $n \times n$ elementary matrices.

3) $\mathbb{R}^1 \cup \mathbb{R}^2$.

4) $\{[1,2], [3,4], [0,0]\}$

4) $\{x \in \mathbb{R}^2 : x_1 = 1\}$.

5) $\{x \in \mathbb{R}^2 : x_2 = 2x_1\}$.

# Easier Way to Tell a Linear Space?

**Is there an easier way to verify a linear space?**

Key property: closed under linear combination

# Subspace

**Definition 11.2 (subspace)**
Suppose $V$ is a linear space.
We say $W$ is a subspace of $V$ if two conditions hold:
i) $W$ is a subset of $V$;
ii) $W$ is a linear space.

In words: A subspace of V is a subset that is itself a linear space.

**Proposition 11.1 (criteria of subspace)**
Suppose $V$ is a linear space. $W$ is a subspace of $V$ if:
i) $W$ is a subset of $V$;
ii) $W$ contains the zero element: $\mathbf{0} \in W$;
iii) $W$ is closed under addition: $\mathbf{u} + \mathbf{v} \in W, \forall\ \mathbf{u}, \mathbf{v} \in W$.
iv) $W$ is closed under scalar multiplication: $\alpha \mathbf{u} \in W, \forall\ \mathbf{u} \in W, \alpha \in \mathbb{R}$.

NOT necessarily the real number 0.

**In words:** A subspace of V is a subset that is closed under linear combination.

# Subspace Example 1

**Eg 4a (biggest subspace)** **V** is a subspace of V.

**Eg 4b** $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Verification:

# Subspace Examples

**Eg 5a** $\{x \in \mathbb{R}^2 : x_2 = 2x_1\}$ is a subspace of $\mathbb{R}^2$.

**Eg 5b** $\{x \in \mathbb{R}^2 : 3x_1 + 5x_2 = 0\}$ is a subspace of $\mathbb{R}^2$.

Geometry: a line in the plane.

**Eg 5c** $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

**Eg 5d** $\{x \in \mathbb{R}^3 : 3x_1 = x_3\}$ is a subspace of $\mathbb{R}^3$.

Geometry: a _____________ in 3D space.

# An Important Subspace

**Eg 5c** $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

**Non-Eg 2** $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 5\}$ is NOT a subspace of $\mathbb{R}^3$.

**Definition** (homogeneous linear equation)
$a_1x_1 + \ldots + a_nx_n = 0$ is a homogeneous linear equation.

**Remark:** $a_1x_1 + \ldots + a_nx_n = b$ where $b \neq 0$ is NOT.

**Fact**: $\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i x_i = 0\}$ is a linear space;

$\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i x_i = b\}$ where $b \neq 0$ is not a linear space.

# Part III Null Space and Column Space

—Null space, or Solution space of Ax = 0
—Span and column space

# Taking intersection?

Revisit the examples.

**Eg 5c** $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

**Eg 5d** $\{x \in \mathbb{R}^3 : 3x_1 = x_3\}$ is a subspace of $\mathbb{R}^3$.

What about the intersection of the two subspaces?

# Taking intersection?

Revisit the examples.

**Eg 5c** $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

**Eg 5d** $\{x \in \mathbb{R}^3 : 3x_1 = x_3\}$ is a subspace of $\mathbb{R}^3$.

What about the intersection of the two subspaces?

Expressed as $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0,\ 3x_1 - x_3 = 0\}$

**Answer**: Still a subspace.

Intuitively, the intersection of two planes is a line.

# Homogeneous Linear System

**Definition 11.2 (homogeneous linear system)**
A homogeneous linear system is

$$Ax = 0$$

where $A \in \mathbb{R}^{m \times n}$ are given and $x \in \mathbb{R}^{n \times 1}$ is the variable

In words: a homogeneous linear system is a linear system with RHS being **0**.

# Homogeneous Linear System

**Definition 11.2 (homogeneous linear system)**
A homogeneous linear system is
$$Ax = 0$$
where $A \in \mathbb{R}^{m \times n}$ are given and $x \in \mathbb{R}^{n \times 1}$ is the variable

In words: a homogeneous linear system is a linear system with RHS being **0**.

Theorem 11.1
The solution set of a homogeneous linear system Ax=0 is a linear space.

Remark: Also a subspace of $\mathbb{R}^n$.

**Definition 11.3** (**null space**)
The solution set of Ax=0 is called the null space of A, denoted as N(A).

# Proof

**Proof of Thm 11.1:**
Denote the solution set as $W$.

Need to verify:
(P1) $0 \in W$.
(P2) W is closed under linear combination.

# Two Ways to Generate Subspaces

Solution set of linear equation +
Taking **intersection**
==> null space

Next, we show another important mechanism of generating a subspace: span (linear combination).

# Definition: Span

**Definition 11.3 (span)**
Suppose $V$ is a linear space.
Suppose $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a subset of V.
The span of $\mathcal{U}$ is defined as

$$\text{span}(\mathcal{U}) \triangleq \{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}),$$

**In words**: **the span (of elements of a linear space)** is the set of all linear combinations of these elements

**Eg**: W=$\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$ is the span of $\{\mathbf{u}, \mathbf{v}\}$ .
**Remark**: For simplicity, we can also say W is the span of $\mathbf{u}, \mathbf{v}$ .

# Definition: Spanning Set

**Definition 11.4 (spanning set)**
Suppose $V$ is a linear space.
Suppose $\mathscr{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a subset of V.
If span$(\mathscr{U})$ = V, then we say $\mathscr{U}$ is a spanning set of V, or $\mathscr{U}$ spans V.

**Eg**: $\{e_1, e_2, \ldots, e_n\}$ is a spanning set of $\mathbb{R}^n$.

**Eg**: $\{\mathbf{u}, \mathbf{v}\}$ is a spanning set of W=$\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$.

Exercise: Find a spanning set of the matrix space.

**Remark**: Spanning set is NOT unique. Cannot say “the spanning set”.

# Definition: Column Space

**Definition 11.3 (column space)**
Suppose $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$. is a linear space.
Then span$(\{\mathbf{a}_1, \ldots, \mathbf{a}_n\})$ is called the column space of A, denoted as C(A).

**In words**: **A's column space is the span of A's column vectors**.

Eg: C($I_n$) = __________

**Eg**: Column space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$ is the set $\left\{ \alpha_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$.

Have you seen this before?

# Column Space and Linear System

**Eg**: Column space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$ is the set $\left\{ \alpha_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$,

or set $\left\{ \begin{bmatrix} \alpha_1 \\ 4\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + 3\alpha_2 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$, or set $\{A\alpha \mid \alpha \in \mathbb{R}^2\}$ .

## Matrix form:

$\mathbf{b} \in C(A) \iff \exists \alpha_1, \alpha_2,$ s.t. $A\alpha = \mathbf{b} \iff A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x}$

## Scalar form:

$\mathbf{b} \in C(A) \iff \exists \alpha_1, \alpha_2,$ s.t. $\begin{bmatrix} \alpha_1 \\ 4\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + 3\alpha_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \iff \begin{cases} \alpha_1 = b_1 \\ 4\alpha_1 + 3\alpha_2 = b_2 \\ 2\alpha_1 + 3\alpha_2 = b_3 \end{cases}$ has at least one solution $(\alpha_1, \alpha_2)$ .

## Column form:

$\mathbf{b} \in C(A) \iff \exists \alpha_1, \alpha_2,$ s.t. ___________ $\iff$ ________ has at least one solution $\mathbf{x}$

# Column space and solvability

**Proposition 11.1**
**Ax=b has a solution iff $b \in C(A)$.**

**The first result** in this class about **solvability** of linear system.
—more will come later.

It a direct outcome of definition:
—b= A x means b is a linear combination of columns of A with coefficients $x_1, x_2, \ldots, x_n$.
—"Solvable" means such a linear combination exists.
—Now we consider set of all linear combinations of columns.
—Thus "solvable" iff b is in this set.

Perspective of looking at "space"
("all possible"; global view).

# Solution Set

**Q1**: “What” is the solution set of A**x**=**b**?

**Partial answer**: When **b** =0, it is a linear space N(A).

**For general** b**?**

**Q2**: How to express/compute the solution set?

# Concluding Part

# Summary Today (write Your Own)

**One sentence summary:**

**Detailed summary:**

# Summary Today (of Instructor)

## One sentence summary:

We study linear space, including null space, column space, span.

## Detailed summary:

1. **Linear Space**

—Set + add, scalar multiply + closed under linear combination + 0,1, negative element
—Euclidean space
—Matrix space; polynomial space

2. **Subspace and span**

—Subspace: subset that is closed under linear combination
—Span: set of linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

3. **Null space and column space**

—Null space N(A) is solution set of Ax = 0
—Column space C(A) consists of linear combinations of columns
—Ax = b solvable iff b in C(A)