Lecture 12

Linear Space II: Subspace, Span and Column Space

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Today's Lecture: Outline

Today … Subspace, span and column space

1. Subspace
2. Span
3. Column space (time permits).

Strang's book: Sec 3.1

Today's Lecture: Learning Goals

After this lecture, you should be able to

1. Verify a linear space by subspace
2. Compute the span of a set; Explain why the span is a subspace
3. Tell the relation of the column space and linear system (time permits)

Part I Linear Space (2)

—Linear space: More Examples

—Subspace

Recall: Motivation and Informal Definition

Solution

What are special about lines, planes?

(Compared to circles, balls, ellipsoids, etc.)

Closed under linear combination

Informally:

Linear space is a set with rules for addition & scalar multiplication:

i) Standard properties of addition and scalar multiplication;

ii) any linear combination of elements is in this space.

closed under LC.

Recall: Euclidean Space and Matrix Space

Eg 1: $\mathbb{R}^n$ is a linear space, called n-dimensional Euclidean space.

Verify Informally: Euclidean space is a linear space, when equipped with addition and scalar-vector product

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_i \in \mathbb{R}, \forall i \right\}.$$

check:

$$\underbrace{\vec{u}, \vec{v}}_{\text{length-}n} \in \mathbb{R}^n \Rightarrow \alpha \vec{u} + \underbrace{\beta \vec{v}}_{\text{length-}n} \in \mathbb{R}^n.$$

Eg 2: $\mathbb{R}^{m\times n}$ is a linear space, called a matrix space.

Verify Informally: Matrix space is a linear space, when equipped with addition and scalar-matrix product

$$\mathbb{R}^{m\times n} = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \middle| \, a_{ij} \in \mathbb{R}, \forall 1 \le i \le m, 1 \le j \le n \right\}.$$

Check.

$$A, B \in \mathbb{R}^{m\times n} \Rightarrow \alpha A + \beta B \in \mathbb{R}^{m\times n}.$$

Example: polynomial space (3rd typical LS).

Eg 3: Set of polynomials with degree no more than k "is" a linear space.

Verification: Polynomial. $f(x)=x^2+2x$ degree $=2$

$f(x)=x-7x^2+x^{21}$ degree $=21$.

f is a polynomial; or $f(x)$ is a polynomial.

$f(y)$ is a polynomial.

$\cancel{\in P_n}$

$\cancel{\sum_i (a_i x^i + b_i x^i)}$.

① Set: $P_2 \triangleq \{ax^2+bx+c \mid a,b,c\in\mathbb{R}\}$; element, $2x^2+7$; x^2-7x+1,

$P_n \triangleq \{\sum_{i=0}^{n} a_i x^i \mid a_i\in\mathbb{R}, \forall i\}$. $[a_0,a_1,\cdots,a_n]$

$f(x)+g(x)$

② Operation: "$+$": If $f(x)=\sum_{i=0}^{n} a_i x^i$, $g(x)=\sum_{i=0}^{n} b_i x^i$, then $(f+g)(x)=\sum_{i=0}^{n}(a_i+b_i)x^i \in P_n$

Scalar product. If $f(x)=\sum_{i=0}^{n} a_i x^i$, $\alpha\in\mathbb{R}$, then $\alpha\cdot f(x)=\sum_{i=0}^{n}(\alpha a_i)x^i \in P_n$

③ LC closed. $f,g\in P_n \Rightarrow f+g\in P_n$

$f\in P_n \Rightarrow \alpha\cdot f\in P_n$.

Non-Example: Polynomial with Exact Degree

$\hat{P}_n \triangleq \{$ polynomials of degree $= n \}$

$\hat{P}_n$ is NOT a linear space.

e.g. $\hat{P}_2 \triangleq \{ ax^2 \mid a \in \mathbb{R},\ a \neq 0 \}$.

① set ✓

② operation, $ax^2 + bx^2 = (a+b)x^2$, Operation well-defined.

$\alpha \cdot a x^2 = (\alpha a)x^2$.

③ LC: $(a+b)x^2 \in \hat{P}_2$ iff $a+b \neq 0$.

Thus $2x^2 + (-2x^2) = 0 \cdot x^2 \notin \hat{P}_2$, not closed under LC.

$\Rightarrow$ NOT a linear space.

Exercise (informal)

Assume the set is equipped with standard addition and scalar multiplication.

Are the following linear spaces? Verify by the informal definition.

set & operation & closed under LC

[Remark:
Informal problems do not appear in hw & exam]

✓ 1) Set of $n \times n$ upper triangular matrices.

(1b) Set of nxn nonzero-diagonal upp triangular.

✗ 2) Set of $n \times n$ elementary matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & & 1 \end{bmatrix}$$

not elementary.

✗ 3) $\mathbb{R}^1 \cup \mathbb{R}^2$. "+". $[1] + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = ?$

✗ 4) $M = \{[1,2], [3,4], [0,0]\}$ $\quad [1,2] + [3,4] = [4,6] \notin M$

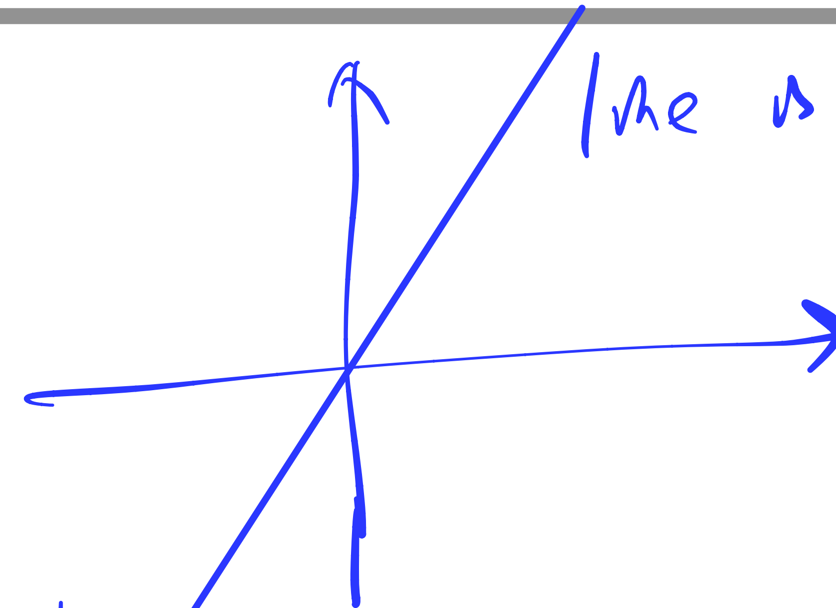
$\in \mathbb{R}^{1\times 2}$ $\quad \in \mathbb{R}^{1\times 2}$

✗ 4) $\{x \in \mathbb{R}^2 : x_1 = 1\}$. $\quad \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\quad 3 \cdot \vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \notin M$. So not closed under LC.

✓ 5) $M = \{x \in \mathbb{R}^2 : x_2 = 2x_1\}$. Verify. ① If $\vec{x} = \begin{bmatrix} t \\ 2t \end{bmatrix} \in M$, $\alpha \cdot \vec{x} = \begin{bmatrix} t \cdot \alpha \\ 2t \cdot \alpha \end{bmatrix} \in M$. ×2

② If $\vec{x} = \begin{bmatrix} t \\ 2t \end{bmatrix}$, $\vec{y} = \begin{bmatrix} s \\ 2s \end{bmatrix} \in M$, $\Rightarrow \vec{x} + \vec{y} = \begin{bmatrix} s+t \\ 2s+2t \end{bmatrix} \in M$.

Linear space inside a linear space?



line is 1D linear space

$\{x \in \mathbb{R}^2 : x_2 = 2x_1\}$.

They are line in a plane.

Linear space inside a linear space.

We call them subspace 子空间.

Subspace: Definition and Verification

Subspace = sub(set) + (linear) space

Definition 11.2 (subspace)
Suppose V is a linear space.
We say W is a subspace of V if two conditions hold:
i) W is a subset of V;
ii) W is a linear space.

In words: A subspace of V is a subset that is itself a linear space.

Subspace: Definition and Verification

Definition 12.1 (subspace)
Suppose V is a linear space.
We say W is a subspace of V if two conditions hold:
i) W is a subset of V;
ii) W is a linear space.

In words: A subspace of V is a subset that is itself a linear space.

[proof skipped]

Proposition 12.1 (criteria of subspace)
Suppose V is a linear space. W is a subspace of V if:
i) W is a subset of V;
ii) W contains the zero element: $\mathbf{0} \in W$;
iii) W is closed under addition: $\mathbf{u} + \mathbf{v} \in W, \forall\ \mathbf{u}, \mathbf{v} \in W$.
iv) W is closed under scalar multiplication: $\alpha \mathbf{u} \in W, \forall\ \mathbf{u} \in W, \alpha \in \mathbb{R}$.

NOT necessarily the real number 0.

subset

$0 \in W$

closed under LC.

Informally: A subspace of V is a subset that is closed under linear combination.

Three ways to verify linear space so far:

① Formal def: 8 rules.
[not required]

② Informal def: set, op, LC closedness.
[textbook] not rigorous

③ Subspace: (a) Linear space's subset W.
(b) LC
(c) $0 \in W$.

Both formal & simple.

Emphasize this method from now on

Rigorous

Remark: Easier Way to Verify Linear Space

Note: Verifying linear space formally is a bit long.

Verifying linear space informally is… informal

If you already have a linear space (often $\mathbb{R}^n$) and a subset, then checking linear space formally is easier.

Key property: closed under linear combination

Extra property: contains 0 element.

Exercise (Formal)

In the following, assume the set is equipped with standard addition and scalar multiplication.

Are the following linear spaces? Verify formally.

$0 \in$

1) ~~Set of $n \times n$ upper~~ triangular matrices.

Subset of $\mathbb{R}^n$ or $\mathbb{R}^{n \times n}$ (linear space)

2) Set of $n \times n$ elementary matrices.

① $0 \in W$

3) $\mathbb{R}^1 \cup \mathbb{R}^2$. not a subset of a known space

② LC.

4) $\{[1,2],[3,4],[0,0]\}$

4) $\{x \in \mathbb{R}^2 : x_1 = 1\}$.

5) $\{x \in \mathbb{R}^2 : x_2 = 2x_1\}$.

$0 \in$

Problem Prove $\{x \in R^2 \mid x_2 = 2x_1\} \triangleq W$ is a linear space.

Proof: ① W is a subset of R^2.

② $\vec{0} \in W$: $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ satisfies $\begin{matrix} 0 = 2 \cdot 0 \\ x_2 = 2 \cdot x_1 \end{matrix}$,

so $\vec{0} \in W$.

③ LC closed:

Verify. ① If $\vec{x} = \begin{bmatrix} t \\ 2t \end{bmatrix} \in M$, $\alpha \cdot \vec{x} = \begin{bmatrix} t \cdot \alpha \\ 2t \cdot \alpha \end{bmatrix} \times 2$ $\in M$.

② If $\vec{x} = \begin{bmatrix} t \\ 2t \end{bmatrix}$, $\vec{y} = \begin{bmatrix} s \\ 2s \end{bmatrix} \in M$, $\Rightarrow \vec{x} + \vec{y} = \begin{bmatrix} s+t \\ 2s+2t \end{bmatrix} \in M$.

Conclusion Thus, by Prop 12.1, W is a subspace of R^2, thus is a linear space.

Subspace Examples

Eg 4a (biggest subspace) **V** is a subspace of V.

Eg 4b $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

$\{\vec{v}\}$ is ___ a subspace

$\vec{v} \neq 0$.

Verification:

① set.

② $\vec{0} \in \{\vec{0}\}$.

③ $\alpha \cdot \vec{0} = \vec{0} \in \{\vec{0}\}$

$\vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\}$.

$\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

① set ✓

② $\vec{0} \notin \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

③ $\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} \notin \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

NOT linear space.

Part II Span

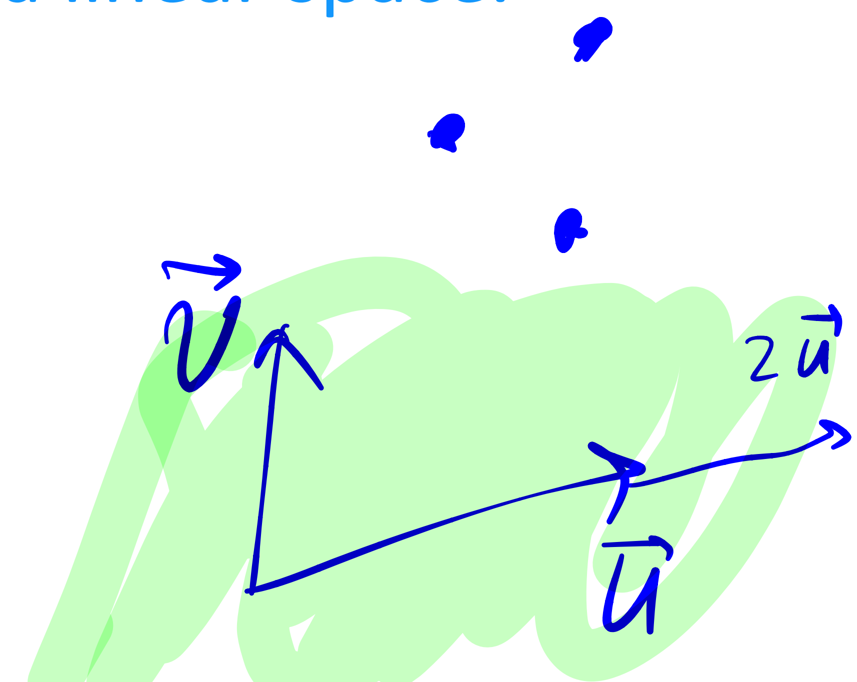
—Sec. 3.1

A Nontrivial Discrete Set is NOT Linear Space

离散

Observation: A nontrivial discrete set is not a linear space.

Trivial discrete set: $\{\mathbf{0}\}$ is a linear space.



Eg1: a single point $\{\mathbf{v}\}$ is NOT a linear space.

Eg2: a set of two points $\{\mathbf{u}, \mathbf{v}\}$ is NOT a linear space.

Motivating Question:

How to expand the set to a linear space?

Motivation: Expanding to Linear Space

Motivating Question:
How to expand a discrete set to a linear space?

Surely, the whole space that contains the set is a linear space.

That's NOT interesting.

An **interesting** question is:

Remark: Identifying a good question
Is extremely important!
In many cases, the question is much more important than answer!

Motivation: Expanding to Linear Space

Motivating Question:
How to expand a discrete set to a linear space?

Surely, the whole space that contains the set is a linear space.

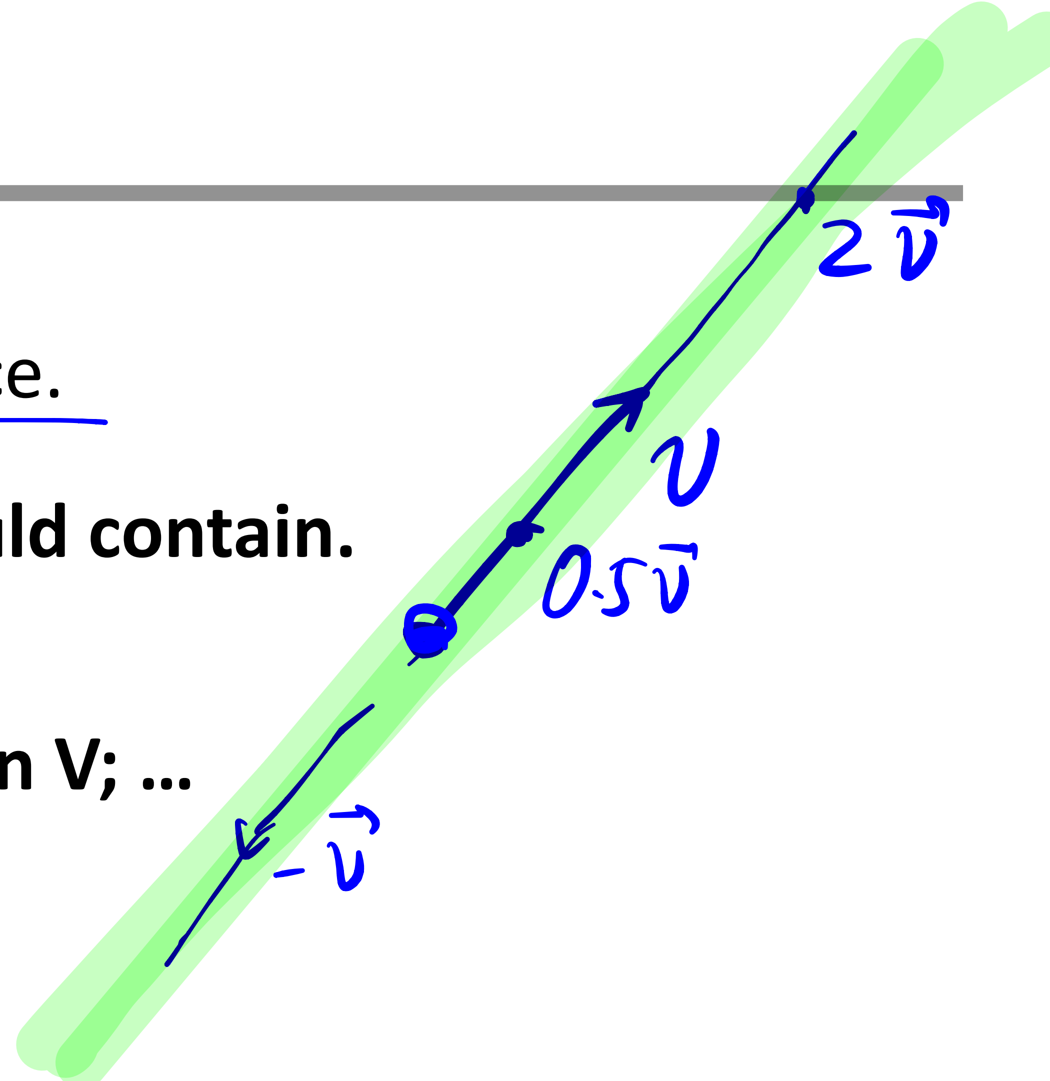
That's NOT interesting.

An **interesting** question is:

What is the minimal linear space that contains $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$?

Remark: Identifying a good question
Is extremely important!
In many cases, the question is much more important than answer!

Expanding One Element



Eg1: a single point $\{\mathbf{v}\}$ $\in R^n$ is NOT a linear space.

Let's analyze what a minimal space V should contain.

First, $2\mathbf{v}$ **should be in V;** $-0.5\mathbf{v}$ **should be in V; …**

In short, $\alpha \vec{v}$ should be in V.

V is at least $M \triangleq \{\alpha \vec{v} \mid \alpha \in R\}$

Is this enough? Is M a linear space?

$> 50\%$ no

$\le 25\%$ yes

Yes.

① subset of R^n.

② $0 \in M$; $0 \cdot \vec{v} \in M$.

③ $\alpha \vec{v} + \beta \vec{v} = (\alpha+\beta)\vec{v} \in M$; $a(\alpha \vec{v}) = (a\alpha)\vec{v} \in M$.

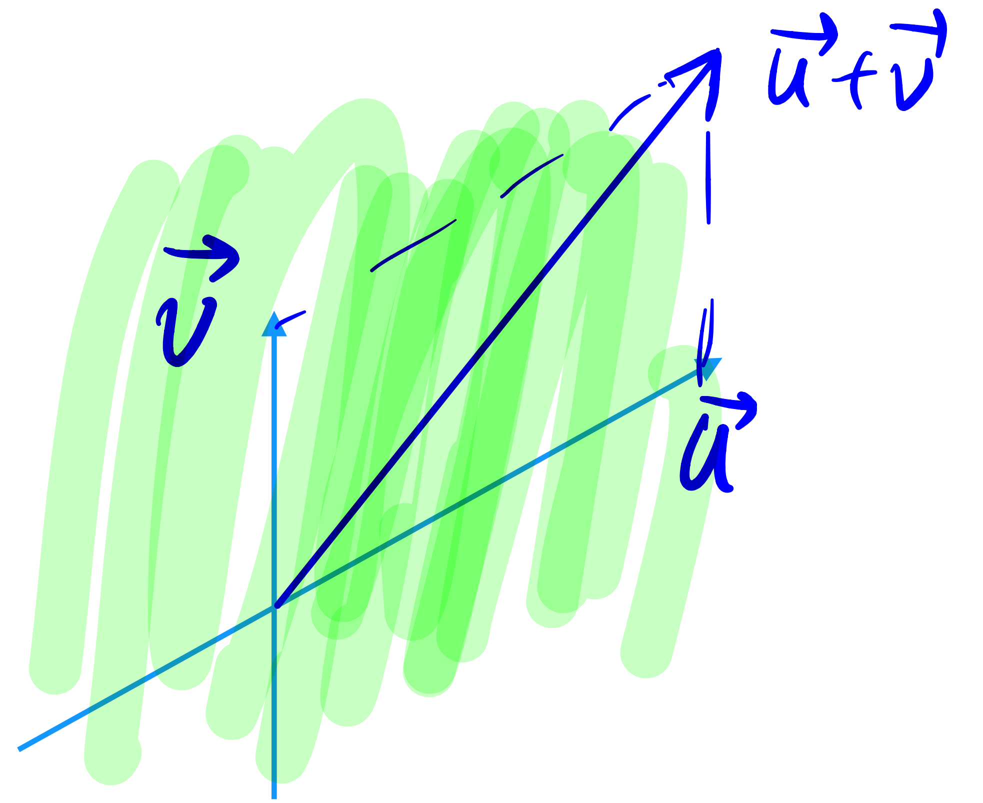
Expanding Two Elements

Eg2: a set of two points $\{\mathbf{u}, \mathbf{v}\}$ is NOT a linear space.

Let's analyze what a minimal space V should contain.

First, $\alpha\mathbf{u}$, $\alpha\mathbf{v}$ should be in V.

V is at least $\{\alpha\vec{u}\} \cup \{\alpha\vec{v}\}$.



Is this enough?

Is V a linear space?

No. Check "+"

collection of 2 lines

No. $\vec{u}+\vec{v} \notin V$.

Expanding Two Elements

Eg2: a set of two points $\{\mathbf{u}, \mathbf{v}\}$ is NOT a linear space.

Let's analyze what a minimal space V should contain.

First, $\alpha\mathbf{u}$, $\alpha\mathbf{v}$ should be in V.

V is at least ____________

Is this enough?

Second, $\alpha\mathbf{u} + \beta\mathbf{v}$ should be in V.

V is at least $M \triangleq \{\alpha\vec{u} + \beta\vec{v} \mid \alpha, \beta \in \mathbb{R}\}$.

Is this enough? i.e. Is M a linear space? Yes.

Yes. Check "$\vec{x}, \vec{y} \in M \Rightarrow \alpha\vec{x} + \beta\vec{y} \in M$."

Key Property: LC of LC is LC

$M \triangleq \{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{R}\}$

Check "$x, y \in M \Rightarrow ax + by \in M$" $\vec{x}, \vec{y} \in M \Rightarrow a\vec{x} + b\vec{y} \in M, \ \forall a, b \in R.$

i.e. "$a(\alpha_1 u + \beta_1 v) + b(\alpha_2 u + \beta_2 v) \in M$" $a(\alpha_1 \vec{u} + \beta_1 \vec{v}) + b(\alpha_2 \vec{u} + \beta_2 \vec{v}) \in M$ want

$(a\alpha_1 + b\alpha_2)\vec{u} + (a\beta_1 + b\beta_2)\vec{v}$ → want: LC of $\vec{u}$ & $\vec{v}$

$p\vec{u} + q\vec{v}$

Key property:
Linear combination of two linear combinations of u, v is a linear combination of u,v.

LC of LC is LC.

transitive

(传递性)

Expanding Any Number of Elements

Eg3: $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is NOT a linear space.

Let's analyze what a minimal space V should contain.

$M \triangleq \{\alpha_1 \vec{u}_1 + \cdots + \alpha_n \vec{u}_n \mid \alpha_i \in \mathbb{R}, \forall i\}$. should be in V.

Is this enough? i.e. Is M a linear space? Yes.

Yes. Check "$\vec{x}, \vec{y} \in M \Rightarrow a\vec{x} + b\vec{y} \in M$"

or $\begin{cases} \vec{x} + \vec{y} \in M \\ \alpha \vec{x} \in M \end{cases}$.

Definition: Span

Definition 11.3 (span)
Suppose V is a linear space.
Suppose $\mathscr{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a subset of V.
The span of $\mathscr{U}$ is defined as

$$\text{span}(\mathscr{U}) \triangleq \{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}),$$

set

In words: **the span (of elements of a linear space)** is the set of all linear combinations of these elements

Fact: The span of any finite subset of V is a subspace of V.

(proved in the last few pages)

Eg: W=$\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$ is the span of $\{\mathbf{u}, \mathbf{v}\}$.
Remark: For simplicity, we can also say W is the span of $\mathbf{u}, \mathbf{v}$.

not rigorous

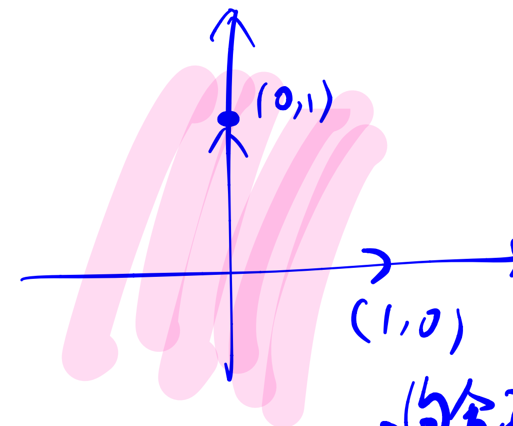
Span of Unit Vectors

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Consider two points (1,0), (0,1) on the plane.

$= \vec{e}_1 \quad = \vec{e}_2$

Span{(1,0), (0,1)} is $\underline{\quad \mathbb{R}^2 \quad}$



Proof: Any element in $\mathbb{R}^2$ can be written as

Important trick

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underset{\text{by Def}}{\in} \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

$\begin{bmatrix} a \\ b \end{bmatrix}$

$\mathbb{R}^2 \subseteq \operatorname{span}(\{\vec{e}_1, \vec{e}_2\})$

$\operatorname{span}(\{\vec{e}_1, \vec{e}_2\}) \subseteq \mathbb{R}^2$

$\Downarrow$

$\operatorname{span}(\{\vec{e}_1, \vec{e}_2\}) = \mathbb{R}^2$.

Claim: span$\{e_1, e_2, \ldots, e_n\} = \mathbb{R}^n$.

Reverse: (formal term defined next page**)**

$\{e_1, e_2, \ldots, e_n\}$ is the $\underline{\quad \text{spanning set} \quad}$ of $\mathbb{R}^n$

$\mathbb{R}^n$: Unit vectors

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\operatorname{span}(\{\vec{e_1}, \ldots, \vec{e_n}\}) = \mathbb{R}^n$$

Implies $\{\vec{e_1}, \ldots, \vec{e_n}\}$ is a "special" set

spanning set

Definition: Spanning Set

Definition 12.2 (spanning set)
Suppose V is a linear space.
Suppose $\mathscr{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a subset of V.
If span$(\mathscr{U})$ = V, then we say $\mathscr{U}$ is a spanning set of V, or $\mathscr{U}$ spans V.

$\mathcal{U} \xrightarrow{\text{span}} V$

spanning set of V

verb.

Eg: $\{e_1, e_2, \ldots, e_n\}$ is a spanning set of $\mathbb{R}^n$.

Eg: $\{\mathbf{u}, \mathbf{v}\}$ is a spanning set of W=$\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$.

Span of Unit Vectors

Exercise: Find a spanning set of the matrix space.
e.g. $\mathbb{R}^{2\times 2}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Remark: Spanning set is NOT unique. Cannot say "the spanning set".

Spanning set: $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Non-spanning set: $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$, does NOT span $\mathbb{R}^{2\times 2}$.

$\mathcal{U}$ span V

$\iff$ $\mathcal{U}$ is a spanning set of V

$\iff$ V = span of $\mathcal{U}$

Terminology

Equivalent statements:

$\{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$ is the span of $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$.

$\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ spans $\{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$

$\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a spanning set of $\{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$

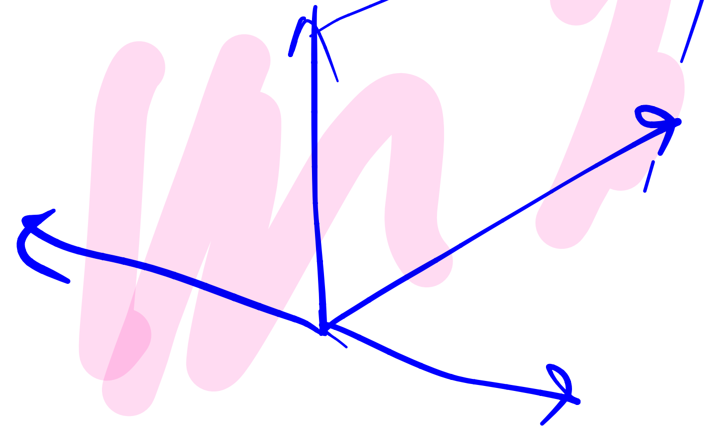
Remark: Spanning set is NOT unique. Cannot say "the spanning set".

Eg: $\{e_1, e_2, \ldots, e_n\}$ is a spanning set of $\mathbb{R}^n$.
$\{e_1, e_2, \ldots, e_n\}$ spans $\mathbb{R}^n$.
$\mathbb{R}^n$ is the span of $\{e_1, e_2, \ldots, e_n\}$.

Intersection and Spanning

Intersection: getting smaller and smaller space

Spanning: Getting larger and larger



Summary Today (write Your Own)

One sentence summary:

Detailed summary:

Summary Today (of Instructor)

Nov. 5

6:30

One sentence summary:

We study subspace, span and column space.

Detailed summary:

1. **Subspace**
   —Verify subspace: subset, closed under LC, contains 0.
   —A single equation can define a subspace [sometimes not]

2. **Span** verb, noun   def. all LC of set
   —Span of a set is the minimal linear space that contains the set
   —Explain why the span is a subspace: LC of LC is LC
   —Span of $\{e_i$'s$\}$ is $\mathbb{R}^n$.

3. **Column space**
   —Def: It is the span of columns of a matrix
   —Ax = b solvable iff $b \in C(A)$.