Lecture 13

Linear Space III: Column Space, Null Space and Solution Set of Ax=b

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Today's Lecture: Outline

Today …

1. Column Space
2. Null Space
3. Computing the solution set of $Ax = b$

Strang's book: Sec 3.3, 3.4

Today's Lecture: Learning Goals

After this lecture, you should be able to

1. Explain relation of column space and solution set

2. Explain relation of null space and solution set

3. Write the general form of solution set of $Ax=b$

4. Verify whether a number of vectors is linearly independent or not

Part 0 Review of Related Contents

Subspace: Definition and Verification

Definition 12.1 (subspace)
Suppose V is a linear space. We say W is a subspace of V if:
W is a subset of V and W is a linear space.

Proposition 12.1 (criteria of subspace)
Suppose V is a linear space. W is a subspace of V if:

i) W is a subset of V;
ii) W contains the zero element: $\mathbf{0} \in W$;
iii) W is closed under addition: $\mathbf{u} + \mathbf{v} \in W, \forall\, \mathbf{u}, \mathbf{v} \in W$.
iv) W is closed under scalar multiplication: $\alpha \mathbf{u} \in W, \forall\, \mathbf{u} \in W, \alpha \in \mathbb{R}$.

NOT necessarily the real number 0.

Subset
$0 \in W$
closed under LC

Informally: A subspace of V is a subset that is closed under linear combination.

Eg 12.1 $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Eg 12.2 $\{x \in \mathbb{R}^2 : x_2 = 2x_1\}$ is a subspace of $\mathbb{R}^2$.

From Lec 12: Span

Definition 12.3 (span)
Suppose V is a linear space.
Suppose $\mathscr{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a subset of V.
The span of $\mathscr{U}$ is defined as span($\mathscr{U}$) $\triangleq \{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$),

Fact: The span of any finite subset of V is a subspace of V.

LC of LC is LC.

Eg 12.3: W=$\{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$ is the span of $\{\mathbf{u}, \mathbf{v}\}$.
Remark: For simplicity, we can also say W is the span of $\mathbf{u}, \mathbf{v}$.

$\text{span}(\{\vec{e}_1, \ldots, \vec{e}_n\}) = \mathbb{R}^n$.

Equivalent statements:

$\{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$ is the span of $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$.

$\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ spans $\{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$

$\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a spanning set of $\{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$

Part I Column Space

related to matrix

Definition: Column Space

Definition 12.3 (column space)
Suppose $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$ is a matrix.
Then $\text{span}(\{\mathbf{a}_1, \ldots, \mathbf{a}_n\})$ is called the column space of A, denoted as C(A).

In words: **A's column space is the span of A's column vectors**.

Eg: $C(I_n) =$ __ R^n __

Eg: Column space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$ is the set $\left\{ \alpha_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$.

Have you seen this before?

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix},$$

$$C(A) = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \middle| \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$

$$= \left\{ \begin{bmatrix} \alpha_1 \\ 4\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + 3\alpha_2 \end{bmatrix} \middle| \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C(I_2) = \underline{\text{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}\right)}$$

$$= \mathbb{R}^2.$$

important truth

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \forall a, b \in \mathbb{R}$$

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix} = [\vec{e}_1, \ldots, \vec{e}_n]$$

$$C(I_n) = \text{span}(\{\vec{e}_1, \ldots, \vec{e}_n\}) = \mathbb{R}^n.$$

Remarks

Column space is "attached to" any matrix A.

Column space of A is defined JUST based on A.

Just like: rows of A, inverse of A are defined based on A.

Column space of A is NOT defined based on linear system Ax=b.

But….

there is a relation…

Column Space and Linear System

Eg: Column space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$ is the set $\left\{ \alpha_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$,

or set $\left\{ \begin{bmatrix} \alpha_1 \\ 4\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + 3\alpha_2 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$, or set $\{A\alpha \mid \alpha \in \mathbb{R}^2\}$.

Matrix form:

$A = [\vec{a}_1, \vec{a}_2]$, $b \in C(A) = \{ \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 \mid \alpha_1, \alpha_2 \in R \}$.

$\mathbf{b} \in C(A)$

$\iff \exists \overset{x_1}{\alpha_1}, \overset{x_2}{\alpha_2}$, s.t. $b = \overset{x_1}{\alpha_1} \vec{a}_1 + \overset{x_2}{\alpha_2} \vec{a}_2$

$\iff$ $b = x_1 \vec{a}_1 + x_2 \cdot \vec{a}_2$ has at least one solution $\mathbf{x}$

column form of linear system

$b = A\vec{x}$ has solution.

Column space and solvability

Proposition 12.1
Ax=b has a solution iff $b \in C(A)$.

Remark: **The first result** in this class about **solvability** of linear system.
—more will come later.

Review the proof:

$\boxed{\textbf{b= A x}} \Longleftrightarrow \boxed{\text{b = LC of columns of A with coefficient x}}$ x,b not specified

def of matrix-vec (column form)

Column space and solvability

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Review the proof:

b= A x	$\Leftrightarrow$	b = LC of columns of A with coefficient x	x,b not specified
b= A x solvable	$\Leftrightarrow$	b = LC of columns of A with coefficient x for some x	Given b Exists x.

Column space and solvability

Proposition 12.1
Ax=b has a solution iff $b \in C(A)$.

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Review the proof:

b= A x	⇔	b = LC of columns of A with coefficient x	x,b not specified
b= A x solvable	⇔	b = LC of columns of A with coefficient x for some x	Given b Exists x.
	⇔	b is in the set {LC of columns of A w coeff x, for all x}	Set: all possible x.

↓ idea of set

Perspective of looking at "space" or "set" ("all possible").

Situation Study: Someone asks you:

is Google a Stanford-created company?

$$b = Ax$$

Company $\overset{\text{formed}}{=}$ Stanford's students $(x_1, x_2, \cdots x_n)$

$b = Ax$ solvable $\iff b \in C(A)$.

For company b, Com find student x that created b	company $b \in$ set of all companies created by CUHK-SZ
For Google, find Stanford graduates Page & Burin who created Google	Google is in the list of companies created by Stanford

Management View

High-level view

$Ax = b_1$

$\vdots$

$Ax = b_{10}$

$\vdots$

$Ax = \hat{b}_1$

$\vdots$

$Ax = \hat{b}_{10}$

$\vdots$

For some b,

$Ax = b$ has a solution

Set of all such $\vec{b}$

$= C(A)$.

For some $\hat{b}$

$Ax = \hat{b}$ does NOT have solution

Part II Null Space

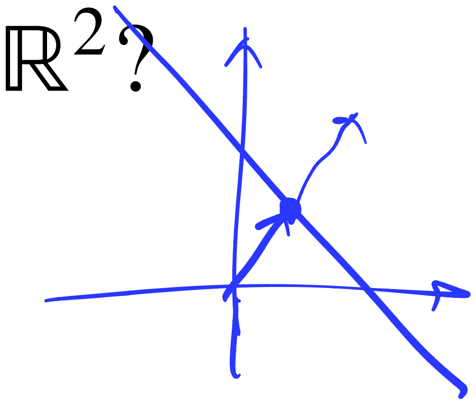
—Sec. 3.2

—Null space, or Solution space of $Ax = 0$

Subspace: Single Equation?

$W \triangleq$

Question: Is $\{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ a subspace of $\mathbb{R}^2$?



No

$\begin{bmatrix}1\\0\end{bmatrix} \in W$

$2\begin{bmatrix}1\\0\end{bmatrix} \notin W$. not closed under LC.

$W \triangleq$

Question: $\{x \in \mathbb{R}^2 : 3x_1 + 5x_2 = 0\}$ is a subspace of $\mathbb{R}^2$.

Yes

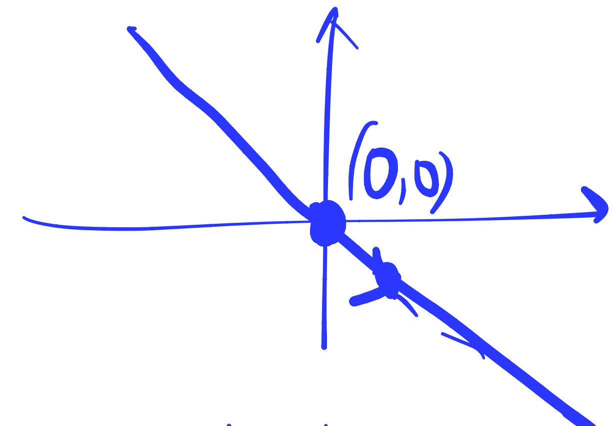
① $0 \in W$

② closed under LC.

If $3x_1 + 5x_2 = 0$

then $3(\alpha x_1) + 5(\alpha x_2) = 0 \Rightarrow (\alpha x_1, \alpha x_2) \in W$

$3(x_1 + y_1) + 5(x_2 + y_2) = 0 \Rightarrow (x_1 + y_1, x_2 + y_2) \in W$



Subspace: Single Equation?

Question: Is $\{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ a subspace of $\mathbb{R}^2$?

Question: Is $\{x \in \mathbb{R}^2 : 3x_1 + 5x_2 = 0\}$ a subspace of $\mathbb{R}^2$.

齐次线性方程.

Definition (homogeneous linear equation)
$a_1x_1 + \ldots + a_nx_n = 0$ is a homogeneous linear equation.

Remark: $a_1x_1 + \ldots + a_nx_n = b$ where $b \neq 0$ is NOT homogeneous.

Fact: $\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_ix_i = 0\}$ is a linear space.

$\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_ix_i = b\}$ where $b \neq 0$ is not a linear space.

Taking intersection? Subspace?

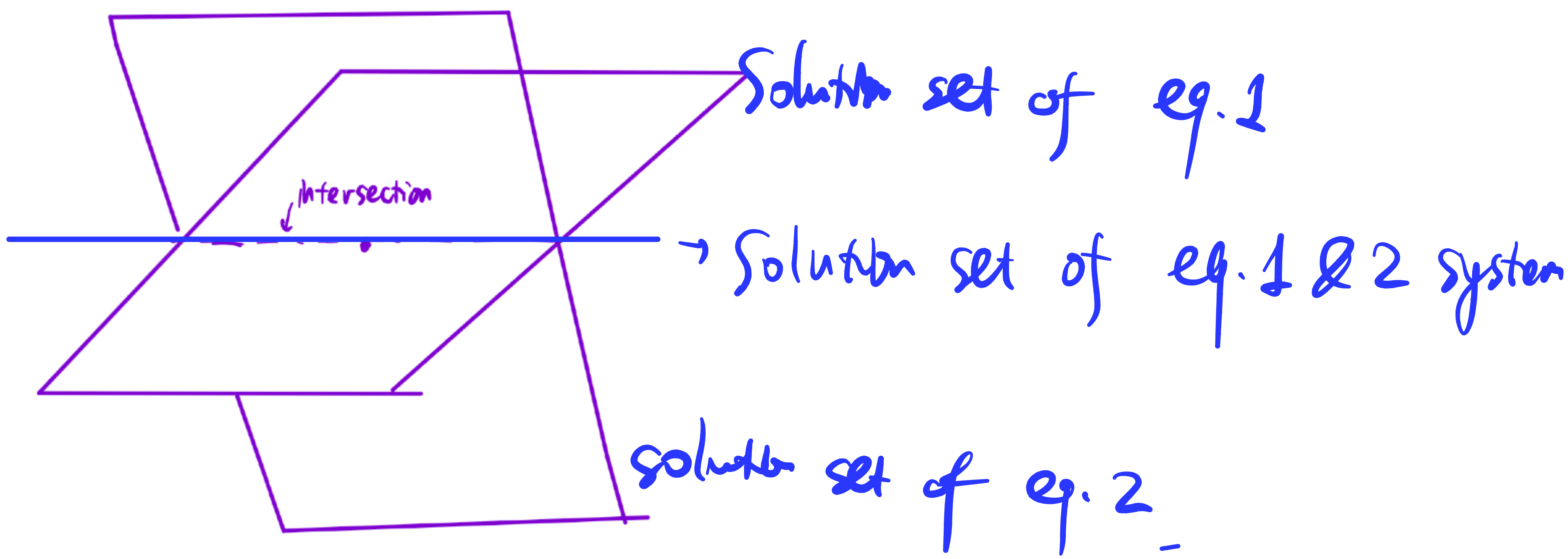
Revisit the examples.

Another mechanism of defining subspaces: intersection.

Eg 5c $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

Eg 5d $\{x \in \mathbb{R}^3 : 3x_1 = x_3\}$ is a subspace of $\mathbb{R}^3$.

What about the intersection of the two subspaces?



Taking intersection?

Revisit the examples.

Eg 5c $\{x \in \mathbb{R}^3 : 3x_1 + 5x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

Eg 5d $\{x \in \mathbb{R}^3 : 3x_1 = x_3\}$ is a subspace of $\mathbb{R}^3$.

What about the intersection of the two subspaces?

Expressed as $\{x \in \mathbb{R}^n : 3x_1 + 5x_2 + x_3 = 0,\ 3x_1 = x_3\}$

Answer: a subspace.

Because, in this example, intersection of two planes is a line.

This geometrical intuition can be generalized to any system.

Homogeneous Linear System

Definition 11.2 (homogeneous linear system)
A homogeneous linear system is

$$Ax = 0$$

where $A \in \mathbb{R}^{m\times n}$ are given and $x \in \mathbb{R}^{n\times 1}$ is the variable

Homogeneous Linear System

Definition 12.1 (homogeneous linear system)
A homogeneous linear system is
$$Ax = 0$$
where $A \in \mathbb{R}^{m \times n}$ are given and $x \in \mathbb{R}^{n \times 1}$ is the variable

In words: a homogeneous linear system is a linear system with RHS being **0**.

Theorem 11.1
The solution set of a homogeneous linear system Ax=0 is a linear space.

Remark: Also a subspace of $\mathbb{R}^n$.

零空间

Definition 11.3 (null space)
The solution set of Ax=0 is called the null space of A, denoted as N(A).

Null(A).

Proof

$W \triangleq \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

Proof of Thm 11.1: Denote the solution set as W.

$W \subseteq \mathbb{R}^n$. ⓪

Step 1. Write definition!!

Need to verify:

(P1) $0 \in W$.

(P2) W is closed under linear combination.

If $\vec{x}, \vec{y} \in W$, $\alpha \in \mathbb{R}$, then $\alpha\vec{x} \in W$, $\vec{x}+\vec{y} \in W$.
(Def of "closed under LC").

Step 2 Verify (P1) and (P2).

Verify (P1) $A \cdot \vec{0} = \vec{0}$, so $\vec{0} \in W$. ①

$\cancel{0 \cdot A = 0}$

Verify (P2). If $\vec{x}, \vec{y} \in W$, $\alpha \in \mathbb{R}$,

then $\underset{m\times n}{A}(\underset{1\times 1}{\alpha}\,\underset{n\times 1}{\vec{x}}) = \underset{1\times 1}{\alpha} \cdot (A\vec{x}) \overset{\text{assume } \vec{x}\in W}{=} \alpha \cdot \vec{0} = \vec{0}$, so $\alpha\vec{x} \in W$ ②

and $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0}+\vec{0} = \vec{0}$, so $\vec{x}+\vec{y} \in W$ ③

Conclusion Combine ⓪, ①, ②, ③, W is a subspace of $\mathbb{R}^n$.

What About Solution Set of Ax = b?

Judgement:

The solution set of a linear system Ax=b is a linear space.

False.

Counter-example: $\{ \vec{x} \in \mathbb{R}^2 \mid x_1 + x_2 = 1 \}$ is NOT a linear space.

$Ax = 0 \longrightarrow$ linear space

Part III Solution Set of $Ax=0$ and $Ax=b$

Strang's book: Sec 3.2, Sec. 3.3

—Expressing null space by span
—Matrix expressions

Revisit: Infinitely Many Solutions for Homogeneous System

RREF

RHS=0.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 4 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot variables

free var.

Equivalent linear system:

$$\begin{cases} x_1 - x_4 + 4x_5 = 0, \\ x_2 + 3x_4 + x_5 = 0, \\ x_3 + 2x_4 + 2x_5 = 0, \\ 0 = 0. \end{cases} \Longrightarrow \begin{cases} x_1 = x_4 - 4x_5, \\ x_2 = -3x_4 - x_5, \\ x_3 = -2x_4 - 2x_5, \\ 0 = 0. \end{cases}$$

s t

Solution set:

$$W = \{[s - 4t, -3s - t, -2s - 2t, s, t]^{\mathsf{T}} \mid s, t \in \mathbb{R}\} \subseteq \mathbb{R}^{5\times 1}$$

Thm 11.1 says: W=Null(A) must be a linear space.

But… do you really "accept" the claim? Or, how to "re-verify" it?

$$W = \left\{ \begin{bmatrix} s-4t \\ -3s-t \\ -2s-t \\ s \\ t \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}. \quad (1)$$

This is the solution set of $Ax=0$, $N(A)=W$. so it's a linear space. But not clear W is a linear space by (1).

How to show W is linear space, in a more direct way?

$$W = \left\{ s \begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}.$$

expression of $N(A)$

which is span of 2 vectors, so linear space.

Re-verify: Solution Set is Linear Space

Solution set:

$$W = \{[s - 4t, -3s - t, -2s - 2t, s, t]^\top \mid s, t \in \mathbb{R}\} \subseteq \mathbb{R}^{5\times 1}$$

Thm 11.1 says: it must be a linear space.
How to "re-verify" it?

Re-verify: Solution Set is Linear Space

Solution set:

$$W = \{[s-4t, -3s-t, -2s-2t, s, t]^{\mathsf{T}} \mid s, t \in \mathbb{R}\} \subseteq \mathbb{R}^{5\times 1}$$

Thm 11.1 says: it must be a linear space.
How to "re-verify" it?

Rewrite:

$$W = \{s\,\underbrace{[1,-3,-2,1,0]^{\mathsf{T}}}_{\mathbf{u}} + t\,\underbrace{[-4,-1,-2,0,1]^{\mathsf{T}}}_{\mathbf{v}} \mid s, t \in \mathbb{R}\}$$

$$= \{s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\} = \mathrm{span}(\{\vec{u}, \vec{v}\}).$$

This is a **set of all linear combinations of two vectors $\mathbf{u}, \mathbf{v}$**.

$\Longrightarrow$ closed under linear combination

$\Longrightarrow$ W is a subspace of $\mathbb{R}^{5\times 1}$

Two ways to describe the solution set of $Ax=0$:

1) solution set of $Ax=0$ is $N(A)$.

2) Solve $Ax=0$, get ($\vec{u}_i$'s can be obtained by RREF).

$$x = \alpha_1 \vec{u}_1 + \cdots + \alpha_q \vec{u}_q$$

Solution set is $\operatorname{span}(\{\vec{u}_1, \cdots \vec{u}_q\})$.

Solution set of Ax = b?

Previous pages: We know how to write the solution set of Ax=0 in terms of linear space.

Next, we consider Ax= b.

It only requires minor extra work.

Example of General System

Augmented matrix in RREF: $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 4 & 1 \\ 0 & 1 & 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 2 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & c \end{array}\right]$

Equivalent linear system:

pivot var   free var.

$$\begin{cases} x_1 - x_4 + 4x_5 = 1, \\ x_2 + 3x_4 + x_5 = 6, \\ x_3 + 2x_4 + 2x_5 = 7, \\ 0 = c\,. \end{cases} \Longrightarrow \begin{cases} x_1 = x_4 - 4x_5 + 1, \\ x_2 = -3x_4 - x_5 + 6, \\ x_3 = -2x_4 - 2x_5 + 7, \\ 0 = c\,. \end{cases}$$

$x_4 \to s$, $x_5 \to t$

Case 1: $c \neq 0$. Solution set is empty.

Case 2: $c = 0$. The solution set is

$W = \{[\alpha - 4\beta + 1, -3\alpha - \beta + 6, -2\alpha - 2\beta + 7, \alpha, \beta]^T \mid \alpha, \beta \in \mathbb{R}\} \subseteq \mathbb{R}^{5\times 1}$

$\{(s-4t+1, -3s-t+6, -2s-2t+7, s, t) \mid s, t \in \mathbb{R}\}$.

Is it a linear space? No.

$$x = (s - 4t + 1,\ -3s - t + 6,\ -2s - 2t + 7,\ s,\ t)$$

$$= s\underbrace{\begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + t\underbrace{\begin{bmatrix} -4 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2} + \underbrace{\begin{bmatrix} 1 \\ 6 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p}$$

Example of General System: Expression By 3 Vectors

$$W = \{[\alpha - 4\beta + 1, -3\alpha - \beta + 6, -2\alpha - 2\beta + 7, \alpha, \beta]^T \mid \alpha, \beta \in \mathbb{R}\} \subseteq \mathbb{R}^{5\times 1}$$

is NOT a linear space.

Each solution can be written as

$$x = \underbrace{\begin{bmatrix}1\\6\\7\\0\\0\end{bmatrix}}_{\vec{x}_p} + \alpha \underbrace{\begin{bmatrix}1\\-3\\-2\\1\\0\end{bmatrix}}_{\vec{v}_1} + \beta \underbrace{\begin{bmatrix}-4\\-1\\-2\\0\\1\end{bmatrix}}_{\vec{v}_2}, \forall \alpha, \beta \in \mathbb{R}.$$

Pick $\alpha = \beta = 0$.

$x = x_p$ is solution.

$$\left[\begin{array}{ccccc|c}1 & 0 & 0 & -1 & 4 & 1\\ 0 & 1 & 0 & 3 & 1 & 6\\ 0 & 0 & 1 & 2 & 2 & 7\\ 0 & 0 & 0 & 0 & 0 & c\end{array}\right]$$

What can we say about the 3 vectors?

$\vec{v}_1, \vec{v}_2$ are solutions of $A\vec{x} = 0$

$\mathbf{x}_p$ is a solution of ? .

Example of General System

Explanation of (P2):

Set $\alpha = \beta = 0$, the resulting vector $\mathbf{x}_p$ is a solution of Ax=b as well.

We call it "a particular solution".

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 1 \\ 6 \\ 7 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -4 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \forall \alpha, \beta \in \mathbb{R}.$$

A complete solution

= (a particular solution) → fixed

+ (any solution of Ax=0). → any vector set

W

$\shortparallel$

x_p vec

$+ \mathcal{N}(A)$ set

Theorem on Solution Set of Ax=b

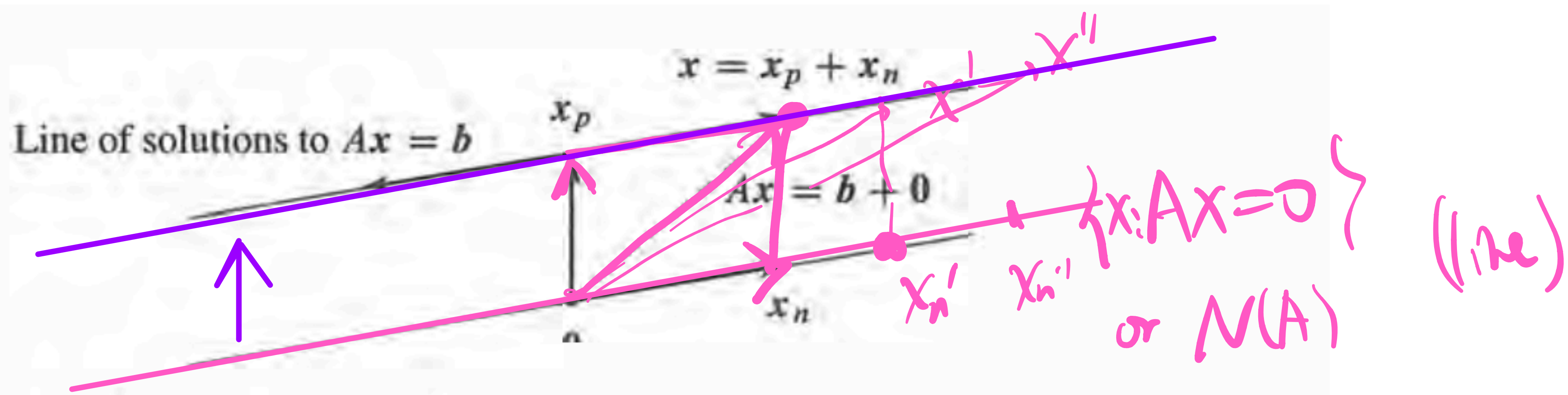
$1 + \{\text{even numbers}\} = \{\text{odd numbers}\}$

Definition: Suppose V is a linear space.
For any element $\mathbf{v} \in V$, and a subspace U, define

$$\mathbf{v} + U \triangleq \{\mathbf{v} + \mathbf{u} \mid \mathbf{u} \in U\}$$

"Shifted" linear space
[not passing origin since $0 \notin W$]

Proposition 13.2
The solution set of a linear system $A\mathbf{x} = \mathbf{b}$ is:
either (i) an empty set
or (ii) $\mathbf{x}_p + N(A)$, where $\mathbf{x}_p$ is any solution of $A\mathbf{x} = \mathbf{b}$.



Complete solution $=$ *one* particular solution $+$ *all* nullspace solutions.

Judgement Questions

① Solution set of linear system can have exactly 2 elements.

② Solution set of linear system can be

$$\{0, \pm 1, \pm 2, \pm 3, \cdots\} = \mathbb{Z}.$$

③ Solution set of linear system can be a circle

$$\{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$$

FALSE

Why false?

Because these sets are NOT shifted linear spaces.

i.e. for any vector $\vec{x_p}$, $W + \vec{x_p}$ is not a linear space.

∴ Prop. 13.2 can help answer questions on the solution set.

Reading: Relation to Inverse

For "good" square linear system A$\mathbf{x}$ = $\mathbf{b}$, the solution is $A^{-1}b$.

For general rectangular system A$\mathbf{x}$ = $\mathbf{b}$, the solution is $\mathbf{x}_p + N(A)$.

Informal derivation:

$B\, x_p + F\, x_F = \vec{b}$,

invertible

$\Rightarrow x_p = \underbrace{B^{-1}\vec{b}}_{x_p} - \underbrace{\overbrace{B^{-1}F}^{C(B^{-1}F) = N(A)}\, x_F}_{x_n}$

How are they related?

Actually, $\mathbf{x}_p = B^{-1}\mathbf{b}_P$

for certain matrix B obtained from A,

and certain vector $\mathbf{b}_P$ obtained from $\mathbf{b}$.

Skip the derivation here.

Think: What Other Questions?

We have presented an algorithm to solve a general linear system of equations.

In short, we are able to solve any linear system now.

What else are NOT known?

Think: What Other Questions?

The complete solution is

$$\mathbf{x}_p + N(A) = \mathbf{x}_p + C(M) = \mathbf{x}_p + \alpha_1\mathbf{v}_1 + \ldots + \mathbf{v}_{n-r}.$$

span$(\vec{v}_1, \ldots, \vec{v}_q)$.

Question: Is there a "simpler" way to express C(M)?

Check an example: $G = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$.

span$\left(\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\2\end{bmatrix}\right\}\right)$

$=$ span$\left(\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}\right)$

How to express C(M) in the simplest way?

span$(\{\vec{u}, 2\vec{u}\})$ $\quad \alpha\vec{u} + \beta \cdot 2\vec{u}$

$=$ span$(\{\vec{u}\})$ $\quad = (\alpha + 2\beta)\vec{u}$.

Part IV Linear Independence

Strang's book Sec. 3.4

Motivation: Simpler Way of Expressing Span?

Observation: $\text{span}(\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\2\end{bmatrix}\}) = \text{span}(\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\})$

Observation: $\text{span}(\{\mathbf{u}, 2\mathbf{u}\}) = \text{span}$ $(\{\vec{u}\})$

Observation: $\text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}\}) = \text{span}$ $(\{\vec{u}, \vec{v}\})$

$\alpha\vec{u} + \beta\vec{v} + \gamma(\vec{u}+\vec{v}) = (\alpha+\gamma)\vec{u} + (\beta+\gamma)\vec{v}.$

Observation: $\text{span}(\{\mathbf{u}, \mathbf{v}, 2\mathbf{u}+\mathbf{v}, 100\mathbf{u}+\mathbf{v}, \mathbf{u}-25\mathbf{v}, 4\mathbf{u}+3\mathbf{v}\})$
$= \text{span}$ $(\{\vec{u}, \vec{v}\})$

$\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4$

Linear Dependence and Independence

线性相关

Definition 13.1 (linear dependence)
Suppose V is a linear space over $\mathbb{R}$.
Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$. We say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear dependent
If there exists real numbers $c_1, c_2, \ldots, c_k$ such that $(c_1, \ldots, c_k) \neq (0,0,\ldots,0)$ and

$$c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k = 0$$

∃ nontrivial combination of u_i's is zero

$\vec{u} + \vec{v} + (-1)\cdot(\vec{u}+\vec{v}) = 0$, $-\vec{u} + \frac{1}{2}(2\vec{u}) = 0$

Think: Can we remove condition $(c_1, \ldots, c_k) \neq (0,0,\ldots,0)$?

opposite

线性无关

Definition 13.2 (linear independence)
Suppose V is a linear space over $\mathbb{R}$.
Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$. We say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent
If for any real numbers $c_1, c_2, \ldots, c_k$ such that $(c_1, \ldots, c_k) \neq (0,0,\ldots,0)$, we have

$$c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k \neq 0$$

any nontrivial LC of u_i's is nonzero

LI

In short, $c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k = 0$ only happens when $c_1 = \ldots = c_k = 0$.

$A \cdot \vec{c} = 0$.

Analogy (formal) "Composition"

$\frac{1}{2}(u+v)$ $\quad$ u $\quad$ v

You, your mom, your dad are linearly dependent.

u $\quad$ v

Your mom, your dad are linearly independent.

$\frac{1}{2}(u+v)$ $\quad$ u

You and your mom are linearly independent

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix} \notin \begin{bmatrix}1\\0\end{bmatrix}$$

Examples

D & I

$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$ are linearly dependent.

$\mathbf{u}, 2\mathbf{u}$ are linearly dependent

$\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ are linearly dependent

$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ are linearly independent

$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\3\\0\end{bmatrix}, \begin{bmatrix}1\\1\\3\\2\end{bmatrix}$ are linearly independent. (require a proof!)

Exercise

D or I ? (Dedependent or Independent)

$$\begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \begin{bmatrix}3\\6\\-3\\-6\end{bmatrix}$$ are linearly ____D____

$\mathbf{u}, 2\mathbf{u}+2\mathbf{v}, 2\mathbf{u}-5\mathbf{v}$ are linearly ____D____

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$ are linearly ____I____

$\mathbf{u}, \mathbf{v}, \mathbf{0}$ are linearly ____D____ (a bit tricky; check by definition)

Columns of $\begin{bmatrix}1 & 0 & a & 0 & c\\0 & 1 & b & 0 & d\\0 & 0 & 0 & 1 & e\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ are linearly ____D____

Linear Independence of Column Vectors

Observation: Columns of matrix A are linearly independent iff $A\vec{x}=\vec{0}$ has unique solution.

For square matrix, Lec 11 says:

(*) $\Longleftrightarrow$ $A^{-1}\ \exists$.

Corollary 13.1: Square matrix A is invertible iff the columns are linearly independent.

$A^{-1}\ \exists$ iff $\vec{a_1}, \dots, \vec{a_n}$ LI.

$A = [a_1, \dots, a_n]$.

Checking Linear Independence

How to Check linear independence?

If the space is $\mathbb{R}^n$, then just need to solve a linear system!

If the space is NOT $\mathbb{R}^n$, need extra tools.

Linear Dependence and Span

Claim 13.1 (linear dependence and span)
Suppose V is a linear space over $\mathbb{R}$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.
If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear dependent, then there exists $t \in \{1, \ldots, k\}$ such that:
i) $\mathbf{u}_t$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_k$.
ii) $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \text{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_k\})$

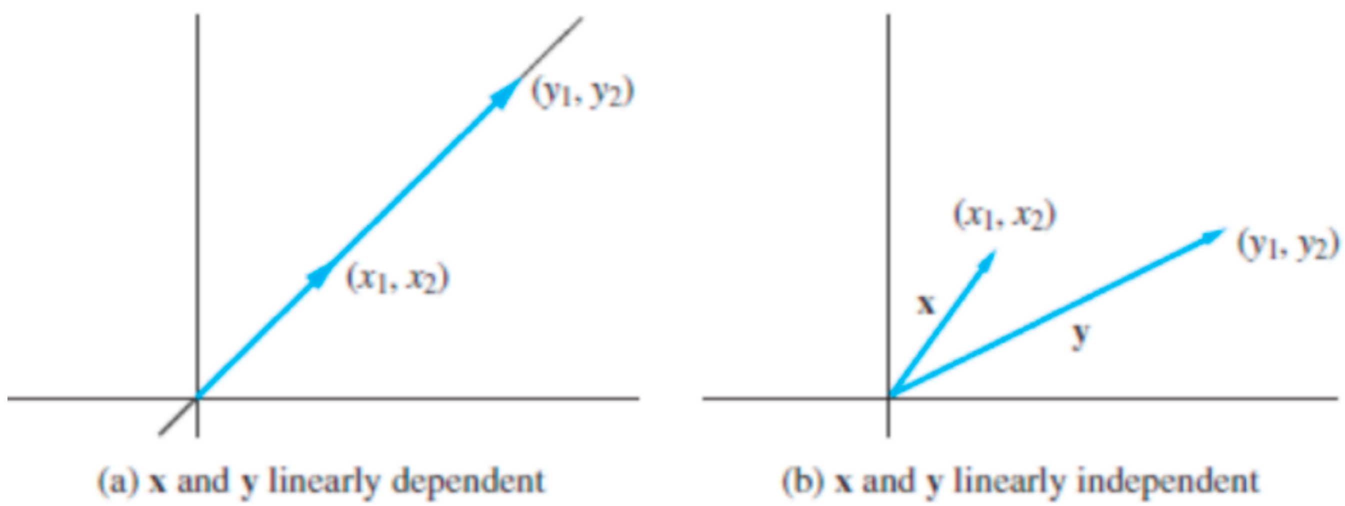
In other words, one element lies in the span of other elements.

The span of these elements can be further simplified.

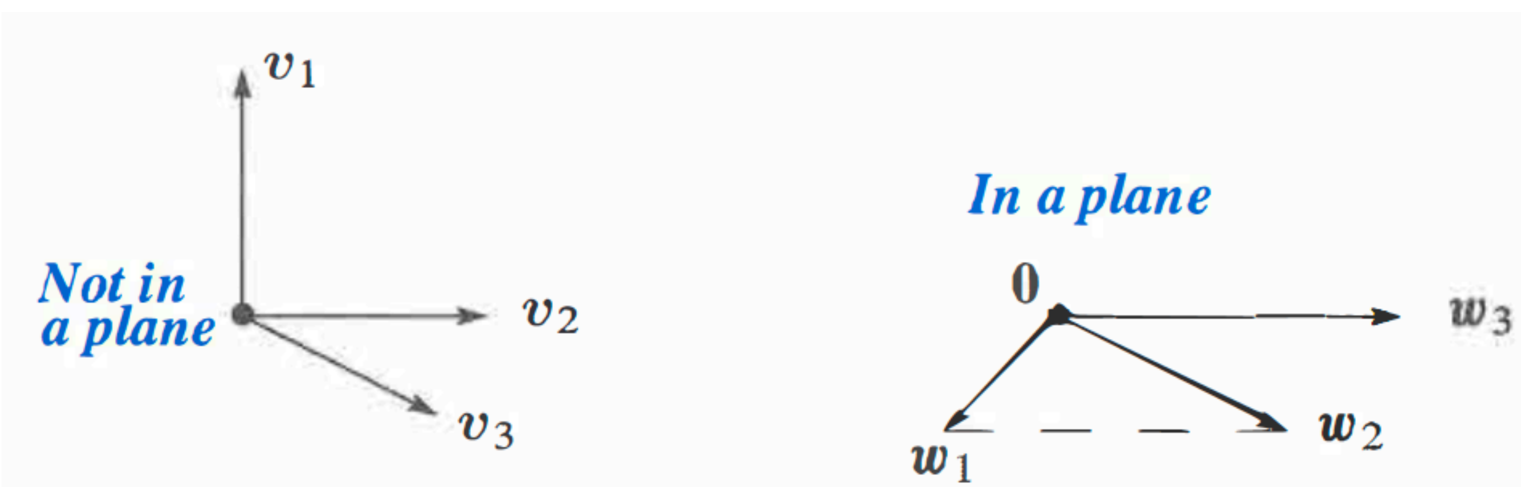
Corollary: If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear independent, then $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ can NOT be simplified (i.e. expressed as the span of $k - 1$ elements)

Geometry

In the plane:



In a 3D space:



Linearly independent vectors

Linearly dependent vectors

Summary Today (write Your Own)

One sentence summary:

Detailed summary:

Summary Today (of Instructor)

One sentence summary:

We study how to solve Ax=b; linear independence, basis and dimension.

Detailed summary:

1. **Solving Ax=b**
   —Solution set of Ax=0: Null space $N(A)$
   —Solution set of Ax=b: $\mathbf{x}_p + N(A)$ or empty set

2. **Linear dependence**
   —Linear dependent elements: trivial linear combination gets 0
   —Related to Ax=0 having infinitely many solutions