Lecture 14

Dimension and Rank

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Today's Lecture: Outline

Today … Dimension, rank

1. Basis and Dimension

2. Rank

Strang's book: Sec 3.3, 3.4

Today's Lecture: Learning Goals

After this lecture, you should be able to

1. Identify basis of common spaces

2. Explain why the dimension of a linear space is a well-defined

3. Compute the number of solutions for different (r,m,n)

Roadmap of Linear Systems

Theme: Solving Linear Systems	Math Tools	What is Learned?	
Part 1 Preparation Lec 3,4,5	Matrix mulplication Row operations	A method to solve any system	Numerical Methods
Part 2 Solving Square system Lec 6,7,8,9,10	LU Decomposition Inverse	"Good" square system: Solution $A^{-1}b$	Analytical Expressions
Part 3 Solving rectangular system Lec 11,12, 13	Linear space Span	Rectangular system: Solution $x_p + N(A)$	
Part 4 Solution set structure Lec 13,14,15	Linear independence Basis Dimension	Solution set dim (n-r) Column space dim r	

Review of Related Contents

From Lec 13: Think: What Other Questions?

The complete solution is

$$\mathbf{x}_p + N(A) = \mathbf{x}_p + \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} .$$

Question: Is there a "simpler" way to express $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$?

$\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent iff the following holds:
$c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k = 0$ only happens when $c_1 = \ldots = c_k = 0$.

General answer: If M's columns are independent, then cannot use fewer vectors to express it.

Part I Basis

Strang's book Sec. 3.4

Linear Dependence and Span

Claim 14.1 (linear dependence and span)
Suppose V is a linear space over $\mathbb{R}$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.
If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear dependent, then there exists $t \in \{1, \ldots, k\}$ such that:
i) $\mathbf{u}_t$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_k$.
ii) span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} =$ span$(\{\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_k\})$

In words:
First, LD <==> one element can be spanned by others.

Second, the **span** of these elements **can be further simplified.**

Corollary: If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear independent, then span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ can NOT be simplified (i.e. expressed as the span of $k-1$ elements)

Motivation: What Spans $\mathbb{R}^n$?

4 unit vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^4$.

Q1: Can we use 3 unit vectors to span $\mathbb{R}^4$?

Q2: Are there 3 vectors that span $\mathbb{R}^4$?

Motivation: What Spans $\mathbb{R}^n$?

5 vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$ span $\mathbb{R}^4$.

Q1: Can we use 4 vectors in these 5 to span $\mathbb{R}^4$?

Q2: Can we use 3 vectors in these 5 to span $\mathbb{R}^4$?

Answer to Q1: No, because they are ____________________

Thus, $\{\mathbf{e}_1, \ldots, \mathbf{e}_4\}$ is not just a spanning set, but also a minimal spanning set (cannot be further simplified).
We will give such spanning set a name: Basis.

Basis

Definition 14.1 (basis)
Suppose V is a linear space over $\mathbb{R}$.
Suppose $\mathscr{U} \triangleq \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} \subseteq V$. We say $\mathscr{U}$ is a basis if
(i) $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent;
(ii) $\text{span}(\mathscr{U}) = V$.

In words, a basis is a linearly independent set and span the whole space.

Eg1 $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$. (called "standard basis")

Eg2 $\{\mathbf{u}\}$ is a basis of span$\{\mathbf{u}\}$, if $\mathbf{u} \neq 0$.

Checking Basis

Preparation: Find a candidate set.

First (independence)
Check the elements are linearly independent.

Second (span)
Check any element can be expressed as a linear combination.

Basis: Exercise

- Find a basis of $\mathbb{R}^{2\times 2}$.

- Find a basis of $P_2 \triangleq \{f(x) = a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$

- Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$?

- Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$?

Basis of Matrix Space

The set B consisting of

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis of $\mathbb{R}^{2\times 2}$. B is linearly independent. Also notice that $\forall A \in \mathbb{R}^{2\times 2}$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aB_{11} + bB_{12} + cB_{21} + dB_{22}.$$

In general, $\{E_{ij}, i = 1,\ldots,m; j = 1,\ldots,n\}$ is a basis of $\mathbb{R}^{m\times n}$,
where E_{ij} is a matrix with only (i, j)-th entry being 1 and other entries
Being 0.

Basis of Polynomial Space

1. $V = P_1$ (polynomials of degree at most 1).

(1) $\mathbf{p}_1(x) = 1$, $\mathbf{p}_2(x) = x$, $\mathbf{p}_3(x) = 2 - 3x$. Is $\mathcal{U} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ a basis for P_1?

No! $\mathcal{U}$ is linearly dependent since $\mathbf{p}_3 = 2\mathbf{p}_1 - 3\mathbf{p}_2$.

(2) $\mathcal{U} = \{1, x\}$ is a basis for P_1.

2. For $V = P_n$ (polynomials of degree at most n),

$\mathcal{U} = \{1, x, x^2, \cdots, x^n\}$ is a basis for P_n.

Part II   Dimension

Strang's book Sec. 3.4

Size of Bases

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$
$\{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$

Question: what is the next interesting quetion?

Size of Bases

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$
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$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$

Question: what is the next interesting quetion?

One answer:

Vote

Q1: What's the smallest basis? **A) 3 B) 2 C) 1**

Q2: What's the biggest basis? **A) ∞ B) 3 C) >3**

Too many vectors cannot be a basis

Claim 14.2a: The set of three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^2$ is a linearly dependent set, thus not a basis of $\mathbb{R}^2$.

Claim 14.2b: columns of $\begin{bmatrix} 1 & 5 & 8 \\ 3 & 8 & 27 \end{bmatrix}$ are linearly dependent.

Equivalent to:
Claim 14.2b' ____________________ has ___________ solution

Proof:

Rewriting the Proof in Shape

Claim 14.2a equivalent to:

Claim 14.3a' ____________________ has ___________ solution

Proof (using shape of matrix, not concrete numbers):

$$\begin{bmatrix} 1 & * & * \\ * & * & * \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * \\ 0 & * & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{bmatrix} \quad \text{Or} \quad \begin{bmatrix} 1 & & \\ 0 & & \end{bmatrix} \quad \text{Or} \quad \begin{bmatrix} 1 & & \\ 0 & & \end{bmatrix}$$

___rows, ___ columns $\Rightarrow$ _________ one ____ variable

$\Rightarrow$ __________ solutions

$\Rightarrow$ Columns of 2×3 matrix are independent

Wide Ax=0 has Non-zero Solution

Equivalent to:

Claim 14.2b ______________________ has ____________ solution

Key Logic Chain of Proof:

∞-many solutions

Free variables

2 pivots, 3 columns

[this example]

# of pivots < # of columns

[in general]

Key Lemma

Claim 14.3: Any $(n+1)$ vectors in $\mathbb{R}^n$ are linearly dependent.

Lemma 14.1
Suppose $m > n$.
If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of a linear space V,
then any m elements $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly dependent.

Unique Representation and Coordinate

Proposition 14.1 (unique representation)
Suppose V is a linear space with a basis $\mathscr{U} \triangleq \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.
Any $\mathbf{v} \in V$ can be written uniquely as a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_n$, namely, there exist unique $[\alpha_1, \ldots, \alpha_n]^\top \in \mathbf{R}^{n\times 1}$ s.t.

$$\mathbf{x} = \alpha_1\mathbf{u}_1 + \ldots + \alpha_n\mathbf{u}_n$$

Definition 14.1 (**coordinates**)
We say $\alpha_1, \ldots, \alpha_n$ are the coordinates of $\mathbf{x}$ with respect to the basis $\mathscr{U}$, denoted as $[\mathbf{x}]_{\mathscr{U}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_n \end{bmatrix}$.

Coordinates of Elements in Matrix Space

Eg. Matrix space.

Coordinates of $\begin{bmatrix} 1 & 5 \\ 3 & 8 \end{bmatrix}$

Dimension = Size of Bases

Theorem 14.1 (bases have same size)
If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are bases of a linear space V, then $m = n$.

First "big" Theorem in this course!

All bases have the same size!! We call it "dimension"!

Proof:

Dimension = Size of Bases

Theorem 14.1 (bases have same size)
If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are bases of a linear space V, then $m = n$.

First "big" Theorem in this course!

All bases have the same size!! We call it "dimension"!

Definition 14.1 (**dimension**)
Suppose V is a linear space。
If V has a basis $\mathscr{U}$ with n elements, then we say the dimension of V is n, denoted as dim(V)=n, or V is n-dimensional.

Eg: dim($\{\mathbf{0}\}$) $= 0$, since $\{\mathbf{0}\}$ has no basis vector.

dim($\mathbb{R}^n$) $= n$, since $\{\mathbf{e}_i, i = 1, \cdots, n\}$ is the standard

dim($\mathbb{R}^{2\times3}$) $= 6$, since basis.

Remark: If there is no finite set of elements that can span V, we say V Is infinite dimensional.

Dimension Measures "Size" of Space

We provide a perspective to understand dimension.

It comes from an old question:

Question: Is a plane "larger" than a line?

You may say: surely larger; but in what sense?

Option 1: # of points.

Option 2: Dimension.

Exercise

What is the dimension of the following linear spaces?

Space of $n \times n$ diagonal triangular matrices

Space of $n \times n$ upper triangular matrices

$$P_n = \left\{ \sum_{i=0}^{n} a_i x^i \,\middle|\, a_i \in \mathbb{R} \right\}$$

$$C\left(\begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

Properties of Basis

A basis is a linearly independent set and span the whole space.

For finite-dimensional linear space:

A basis is a minimal spanning set
(deleting a vector from the basis cannot span the whole space),

Also a maximal independent set
(adding a vector to the basis makes the set linearly dependent)

Quicker Way to Check Basis

Proposition 14.2
Consider a linear space V with dim(V) = n.
If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent, then they form a basis of V.
If $\mathbf{u}_1, \ldots, \mathbf{u}_n \in V$ can span V, then they form a basis of V.

Remark: For n vectors, only need to check either "span" or "lin. indep.", not both.

E.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}$ are linearly independent, thus a basis of $\mathbb{R}^3$.

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Recall: Lec 13:
$n \times n$ matrix A is invertible iff the n columns are linearly independent.

From Prop. 14.2, this is equivalent to:
a) Its columns form a basis.
b) Its columns span the whole space, i.e., Ax=b is solvable for any b.

Invertibility Conditions

Theorem 14.2a (Equivalent Conditions for Invertibility)

Let $A \in \mathbb{R}^{n \times n}$.

The following statements are equivalent:

1. A is invertible
2. The linear system $A\mathbf{x} = 0$ has a unique solution $\mathbf{x} = \mathbf{0}$
3. A is a product of elementary matrices
4. A has n pivots; or equivalently: rank(A) = n
5. The columns of A span $\mathbb{R}^n$.
6. The columns of A are linearly independent;
7. The columns of A form a basis;
8. dim(C(A)) = n.
9. Ax=b is solvable for any b

Equation solving

Rank

Span

Linear independence

Basis

Dimension

Equation solving

Most major concepts we learned so far are related to invertibility.

Part III Rank

Strang's book Sec. 3.4

Rank

The number of pivots r is important.

It is so important that we give it a name.

Definition 14.3 (rank)
The rank of a matrix A is the number of pivots, denoted as rank(A).

Remark: This is a computational definition.

In the future, we will introduce equivalent definitions that are more fundamental.

Property: For $m \times n$ matrix, rank(A) $\leq \min\{m, n\}$.

of Solutions for Different Cases of m, n and r

After proper re-ordering of columns, any RREF can be written as

$$\begin{bmatrix} I_r & F \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} x_P \\ x_F \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}}_{r\times 1} \\ \mathbf{c}_{(m-r)\times 1} \end{bmatrix}.$$

Case 1: $r < m$: [there are zero rows]

Dependent on how large n,m,r, some of the blocks may disappear.

IF $\mathbf{c}_{(m-r)\times 1} \neq \mathbf{0}$: [get 0 = non-zero]

REPORT No solution.

ELSE [get 0 = 0 equations]

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ELSE [get 0 = 0 equations]

IF $r = n$: [no free column] → $\begin{bmatrix} I_r \\ \mathbf{0}_{(m-r)\times r} \end{bmatrix}$

REPORT: 1 solution

ELSE: [there are free columns] → $\begin{bmatrix} I_r & F \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix}$

REPORT: ∞ solutions.

of Solutions for Different Cases of m, n and r

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REPORT No solution.

ELSE [get 0 = 0 equations]

IF $r = n$: [no free column] → $\begin{bmatrix} I_r \\ \mathbf{0}_{(m-r)\times r} \end{bmatrix}$

REPORT: 1 solution

ELSE: [there are free columns] → $\begin{bmatrix} I_r & F \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix}$

REPORT: ∞ solutions.

Case 2: $r = m$: [No zero rows]

IF $r = n$: [no free column] → $\begin{bmatrix} I_r \end{bmatrix}$

REPORT: 1 solution

ELSE: [there are free columns] → $\begin{bmatrix} I_r & F \end{bmatrix}$

REPORT: ∞ solutions.

of Solutions for Different Cases of m, n and r

After proper re-ordering of columns, any RREF can be written as

$$\begin{bmatrix} I_r & F \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} x_P \\ x_F \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}}_{r\times 1} \\ \mathbf{c}_{(m-r)\times 1} \end{bmatrix}.$$

Dependent on how large n,m,r, some of the blocks may disappear.

Four types for R	$\begin{bmatrix} I \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$r = m = n$	$r = m < n$	$r = n < m$	$r < m, r < n$
# of solutions	1	∞	0 or 1	0 or ∞
# of solutions when $\mathbf{c}_{(m-r)\times 1} = \mathbf{0}$	1	∞	1	∞

Relation of Dimension and Rank

Thm 14.2 implies: rank(A) = n <==> dim(C(A)) = n.

Let's recall what it means.

rank(A) = n <==> n pivots [Def of rank]
<==> Ax=0 has unique solution 0 [row transf. does not change solution]
i.e. Ax=0 has no non-zero solution
<==> n columns are linearly independent [Def of lin. ind.]
<==> C(A) has dimension n [Fact 14.1; Def of dim]

Anything interesting to guess?

Guess: rank(A) = dim(C(A)), for any A.

Definition: Row Space

Definition 14.2 (row space)

Suppose $A = \begin{bmatrix} \mathbf{a}_{(1)}^\top \\ \cdots \\ \mathbf{a}_{(m)}^\top \end{bmatrix} \in \mathbb{R}^{m\times n}$. is a linear space.

Then $\text{span}(\{\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(m)}\}) \subseteq \mathbb{R}^n$ is called the row space of A, denoted as **Row(A)**.

In words: **A's row space is the span of A's row vectors**.

Remark: Row(A) = C($A^\top$).

Eg: C(I_n) = ____________

Eg: Row space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$ is the set

$$\left\{\alpha_1\,[1,0]^\top + \alpha_2\,[4,3]^\top + \alpha_3\,[2,3]^\top \mid \alpha_1, \alpha_2 \in \mathbb{R}\right\}.$$

What is dim(Row(R))?

Consider the RREF R.

Pivot rows

Non-pivot rows

$$\begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{0}_{1\times 4} \end{bmatrix}.$$

Observation 1: Span{rows} = span { pivot rows}.

Observation 2: Pivot rows are linearly independent.

Why?

Together: dim(Row(R)) = # of pivots = r.

Rank Theorem

Definition 14.3 (**row-rank and column rank**)
Row-rank of A is defined as dim(Row(A)), denoted as $r_R(A)$.
Column-rank of A is defined as dim(C(A)), denoted as $r_C(A)$.

Theorem 14.3
Row-rank, column-rank and rank of a matrix are the same, i.e.,

$$r_R(A) = r_C(A) = \text{rank}(A).$$

Equivalent expression: **dim(Row(A)) = dim(C(A)) = # of pivots.**

Example

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{Row}(A) = \textbf{Span}\left([1,2,0,0]^T, [0,0,1,0]^T, [0,0,0,1]^T\right)$$

$r_R(A) = 3.$

$$R = \begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \textbf{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}\right)$$

$r_C(A) = 3.$

3 pivots,
So rank(A) = 3.

Thus, $r_R(A)$ **=** $r_C(A)$ **= rank(A).**

Concluding Part

Strang's book Sec. 3.4

How to Fill the Gap Between Courses and Homework & Exam?

First, figure out all logic & proofs of the lectures

Second, work on more practice problems.

—Different from high school:

High school: Teachers (almost) provide you all needed problems.

University: You need to seek more practice problems yourself.
[Active learning]

—Where? Textbook practice problems.
—Too many? Pick a few to work on. It is part of the challenge.

Summary Today (write Your Own)

One sentence summary:

Detailed summary:

Summary Today (of Instructor)

One sentence summary:

Dimension, Rank and Nullity.

Detailed summary:

1. **Basis and Dimension**
   —Dimension of linear space = size of any basis.
   —Theorem 14.1: Any basis has the same size (size = # of vectors in the basis).

2. **Rank**
   Column/row-rank = dim of column/row space
   Rank theorem: row-rank = column-rank = rank (= # of pivots)

3. **Nullity**
   —Nullity = dimension of null space.
   —Nullity theorem: Nullity + rank = n.

Byproduct: Conditions for checking invertibility by
dim, rank, nullity, linear independence.

Appendix: Proof of Rank = Row Rank = Column Rank

Revisiting Solution Set of Ax=0

Pivot columns

Pivot variables

Linear system:

$$\begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Free columns

Free variables

The complete solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \begin{bmatrix} -a \\ -b \\ \mathbf{1} \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -c \\ -d \\ 0 \\ -e \\ \mathbf{1} \end{bmatrix}, \forall\, \alpha, \beta \in \mathbb{R}.$$

Plugging in solutions:

$$\begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -a \\ -b \\ \mathbf{1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow (-a)\mathbf{v}_1 + (-b)\mathbf{v}_2 + \mathbf{v}_3 = 0.$$

$$\begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -c \\ -d \\ 0 \\ -e \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow (-c)\mathbf{v}_1 + (-d)\mathbf{v}_2 + (-e)\mathbf{v}_4 + \mathbf{v}_5 = 0.$$

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can express $\mathbf{v}_4, \mathbf{v}_5$.
They are linearly independent.

Column Space Dim = Rank

Fact 1: In RREF, free columns can be expressed as LC of pivot columns.

Fact 2: **dim(C(R))** = # of pivot columns = **r.**

What about dim(C(A))?

dim(C(A)) = r.

Guess: **dim(C(A)) = dim(C(R)).**

Lemma 14.2 Elementary row operations do not change dimension of column space.

Elementary Row Operation does Not Change LC Relation

Pivot columns

$$\begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -a \\ -b \\ \mathbf{1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Free columns

$$\Longleftrightarrow (-c)\mathbf{v}_1 + (-d)\mathbf{v}_2 + (-e)\mathbf{v}_4 + \mathbf{v}_5 = 0.$$

Consider the matrix $\hat{R}$ that becomes R after one step of elementary row operation.

$$\hat{R} = \begin{bmatrix} 1 & 0 & a & -\beta & c-\beta e \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\beta R_3 + R_1} \begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

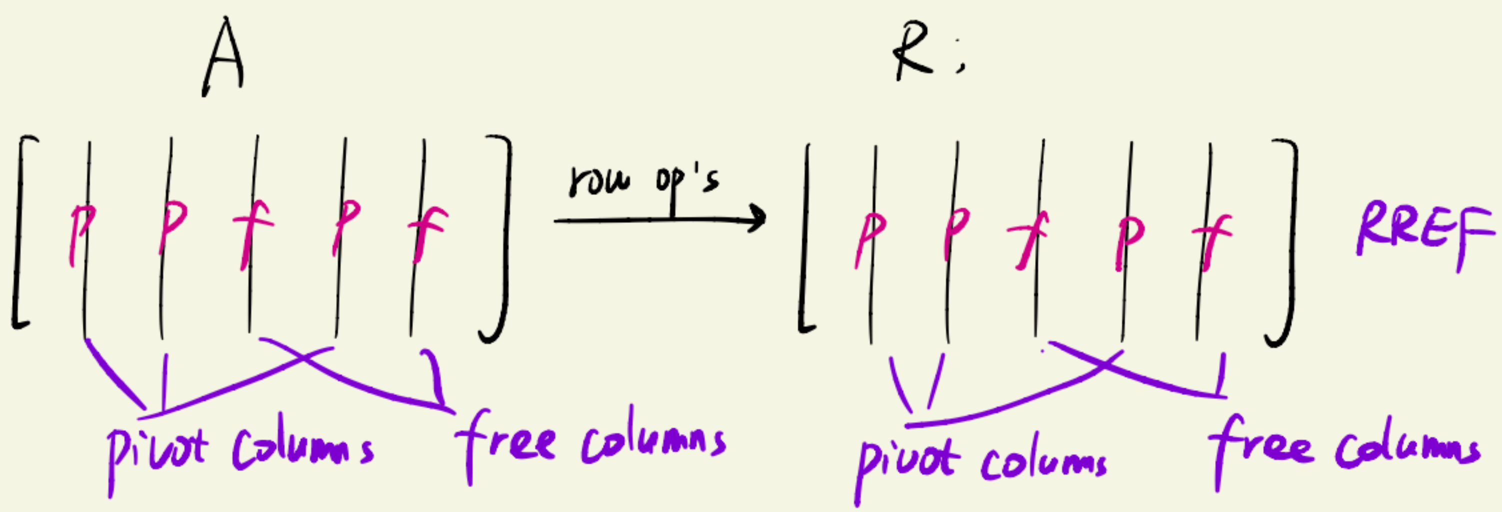
Check: $(-c)\mathbf{v}_1 + (-d)\mathbf{v}_2 + (-e)\hat{\mathbf{v}}_4 + \hat{\mathbf{v}}_5 =$

Claim 14.1: Elementary row operation does not change "linear combination" relation.

i.e. if $\alpha_1 \cdot (\text{Column } 1) + \ldots + \alpha_n \cdot (\text{Column } n) = 0$ holds for M, then it holds for the row-transformed matrix too.

Finishing Proof

to prove $\dim(C(A)) = \dim(C(R)) = r$.



pivot columns of A span free columns of A $\xLeftarrow{\text{Claim 14.1}}$ pivot columns of R span free columns of R

pivot columns of A are lin. independent $\xLeftarrow{\text{Claim 4.1}}$ pivot columns of R are lin. independent

r pivot cols of A form a basis of $C(A)$

$\Downarrow$

$\dim(C(A)) = r$.

Elementary Row Operations do Not Change Row Space

Consider the matrix $\hat{R}$ that becomes R after one step of elementary row operation.

$$W = \begin{bmatrix} \mathbf{w}_1^\top \\ \mathbf{w}_2^\top \\ \mathbf{w}_3^\top \\ \mathbf{w}_4^\top \end{bmatrix} \xrightarrow{\beta R_3 + R_1} \begin{bmatrix} \beta \mathbf{w}_3^\top + \mathbf{w}_1^\top \\ \mathbf{w}_2^\top \\ \mathbf{w}_3^\top \\ \mathbf{w}_4^\top \end{bmatrix} = \hat{W}.$$

Row(W) = span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$

span$\{\beta\mathbf{w}_3 + \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$=Row($W$).

Claim 14.2: Elementary row operation does not change the row space.

dim(Row(A)) = r.

dim(Row(R)) = # of pivots = r