Lecture 15

Four Fundamental Subspaces

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Today's Lecture: Outline

Today … Four Fundamental Subspaces

1. Full-rank
2. Nullity
3. Orthogonal complement
4. Four fundamental subspaces

Strang's book: Sec 3.4, 3.5

Today's Lecture: Learning Goals

After this lecture, you should be able to

1. Tell the # of solutions of full-row-rank, full-column-rank matrices
2. Judge orthogonal complement of another subspace
3. Tell four fundamental subspaces and their relations

Review of Related Contents

Review of Linear Independence and Basis

$\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent iff the following holds:
$c_1\mathbf{u}_1 + \ldots + c_k\mathbf{u}_k = 0$ only happens when $c_1 = \ldots = c_k = 0$.

$\mathbf{u}_1, \ldots, \mathbf{u}_n$ form a basis of V iff the following two hold:

i) they are linearly independent; [no redundant information; not too many]

ii) they span V. [no loss of information; not too few]

Eg1 $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$. (called "standard basis")

Eg2 $\{E_{ij}, i = 1,\ldots,m; j = 1,\ldots,n\}$ is a basis of $\mathbb{R}^{m\times n}$.

Review of Dimension and Rank

Theorem 14.1 (bases have same size)
If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are bases of a linear space V, then $m = n$.

First big supporting Theorem in this course!

All bases have the same size!! We call it "dimension"!

Theorem 14.3
Row-rank, column-rank and rank of a matrix are the same, i.e.,

$$r_R(A) = r_C(A) = \textbf{rank(A).}$$

Second big supporting Theorem in this course!

Equivalent expression: **dim(Row(A)) = dim(C(A)) = # of pivots.**

Supporting theorems:
They are so basic that you will take them for granted after 1 year, or even forget they are "theorems"!

Disclaimer: Maybe "PA=LU" is another "big" theorem, but not a "supporting"-type theorem.

Part I Full Rank Matrices

Full Rank Matrix

Definition 14.4 (full rank matrix)
Let $A \in \mathbb{R}^{m \times n}$.
If rank(A) = m, then we say A has full row rank. 行满秩,
If rank(A) = n, then we say A has full column rank. 列满秩,
If rank(A) = $\min\{m, n\}$, then we say A has full rank. 满秩,

# of rows

Remark: Can also say "A is a full rank matrix".

RREF

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 5 \\ 3 & 3 & 5 \end{bmatrix} \xrightarrow{\textit{Elementary Row operations}} \overbrace{\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}}^{3 \text{ col's}}$$

4 rows

3 pivots

Is this full rank, full row rank, full column rank?

✓ × ✓

Row-rank and Column-rank

Recall: $m - r, n - r$ are critical for # of solutions.

$\begin{bmatrix} 1 & F \\ 0 & 0 \end{bmatrix}$

$m - r$ shall be interpreted as (# of rows) - (row rank)

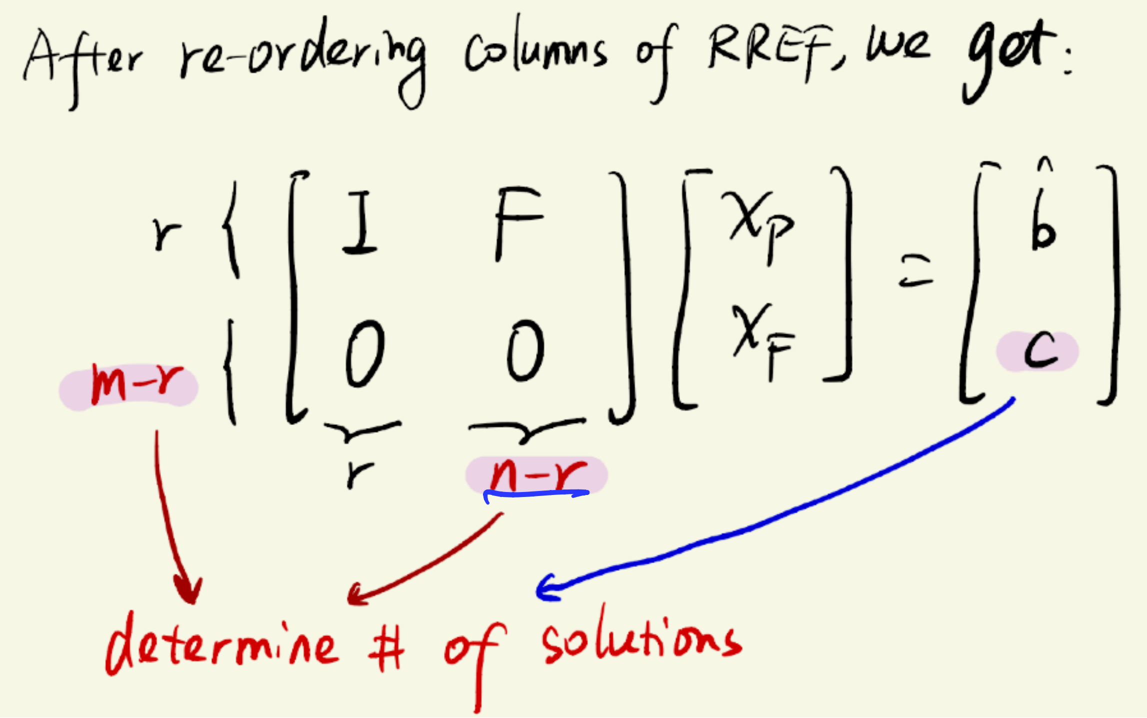
$n - r$ shall be interpreted as (# of columns) - (col rank)

Think: What's special about "full row rank" and "full col rank" matrix?

Solution set!

From Lec 13: m-r and n-r

Lec 14 review & new concept (full rank)



After re-ordering columns of RREF, we get:

$$\begin{array}{r} r\{ \\ m-r\{ \end{array}\begin{bmatrix} I & F \\ \underbrace{0}_{r} & \underbrace{0}_{n-r} \end{bmatrix}\begin{bmatrix} x_p \\ x_F \end{bmatrix} = \begin{bmatrix} \hat{b} \\ c \end{bmatrix}$$

determine # of solutions

If full row rank

$m=r$

$m-r=0$

eliminate 0 rows

no possibility of contradiction

no possibility of 0 solution

1 or ∞

$$m\begin{bmatrix} I & \overset{n-r}{F} \end{bmatrix}$$

not full col rank

∞

full col rank

$[I_r]$

1

If full col rank

$n=r$, or $n-r=0$

$$\begin{bmatrix} I \\ 0 \end{bmatrix}\}m-r$$

full row

$[I]$

1

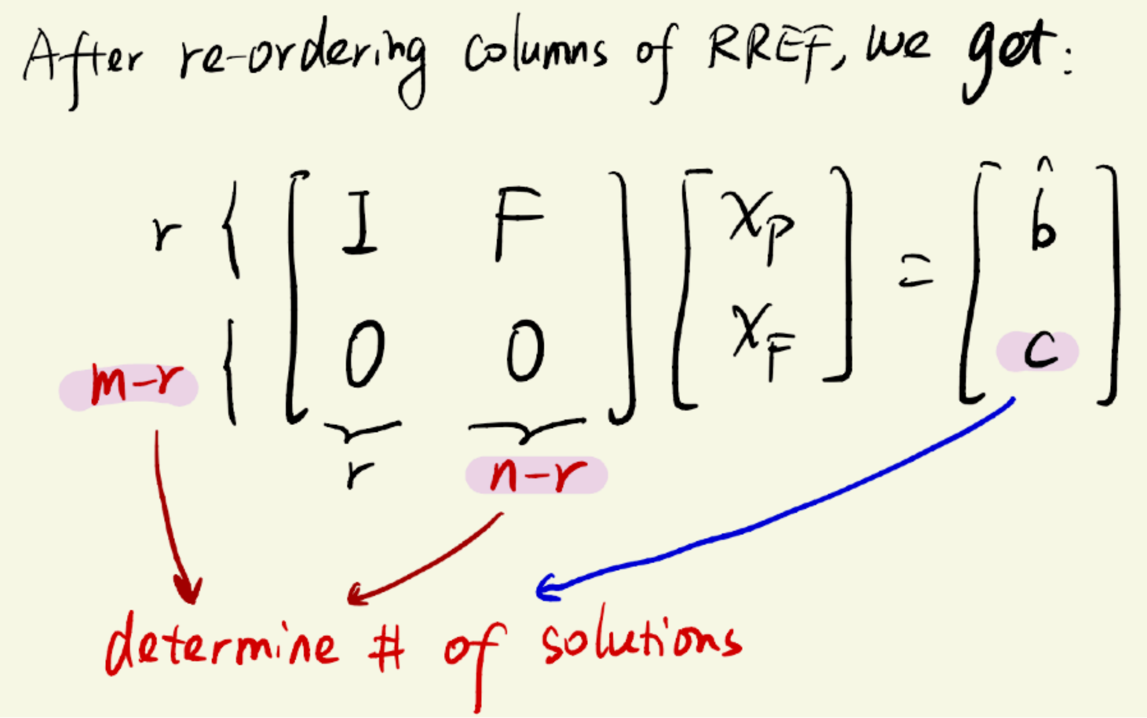
0 or 1

eliminate F

no possibility of ∞ sol

0 or 1

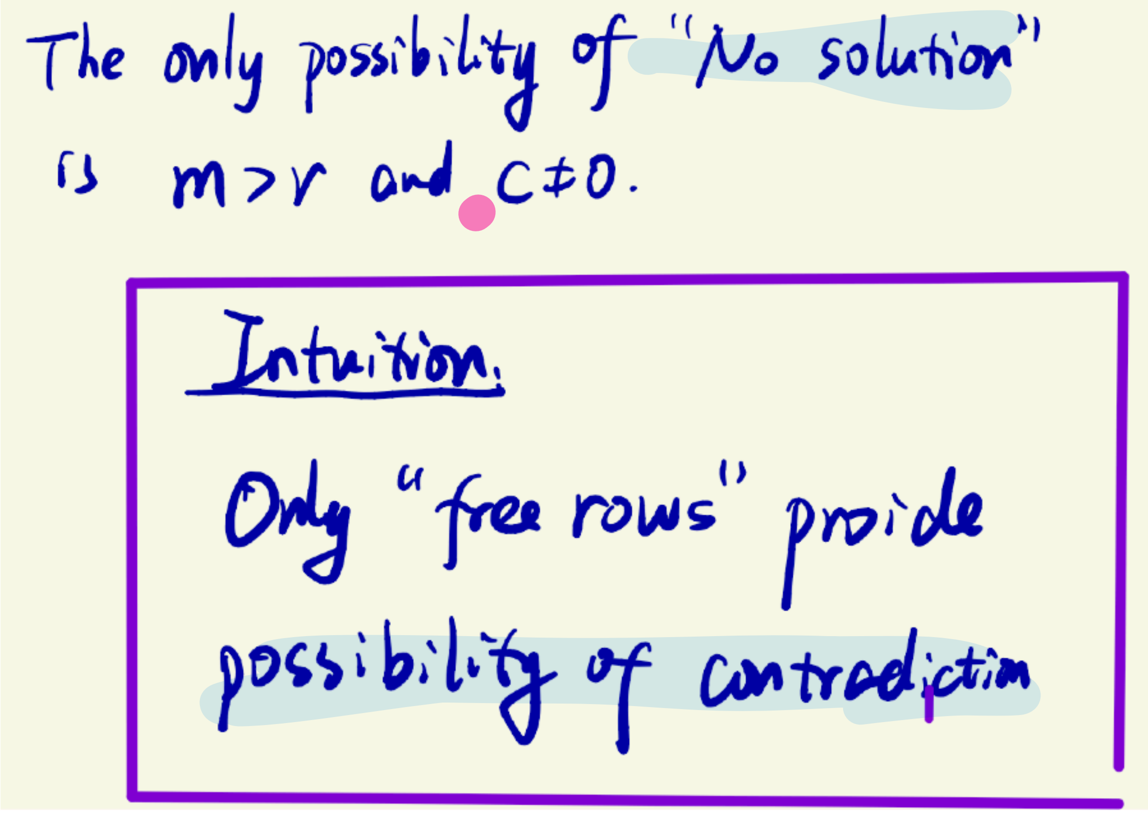
From Lec 13: m-r and n-r



After re-ordering columns of RREF, we get:

$$\begin{array}{c} r\left\{\vphantom{I}\right. \\ m-r\left\{\vphantom{0}\right. \end{array}\begin{bmatrix} I & F \\ \underbrace{0}_{r} & \underbrace{0}_{n-r} \end{bmatrix}\begin{bmatrix} x_P \\ x_F \end{bmatrix} = \begin{bmatrix} \hat{b} \\ c \end{bmatrix}$$

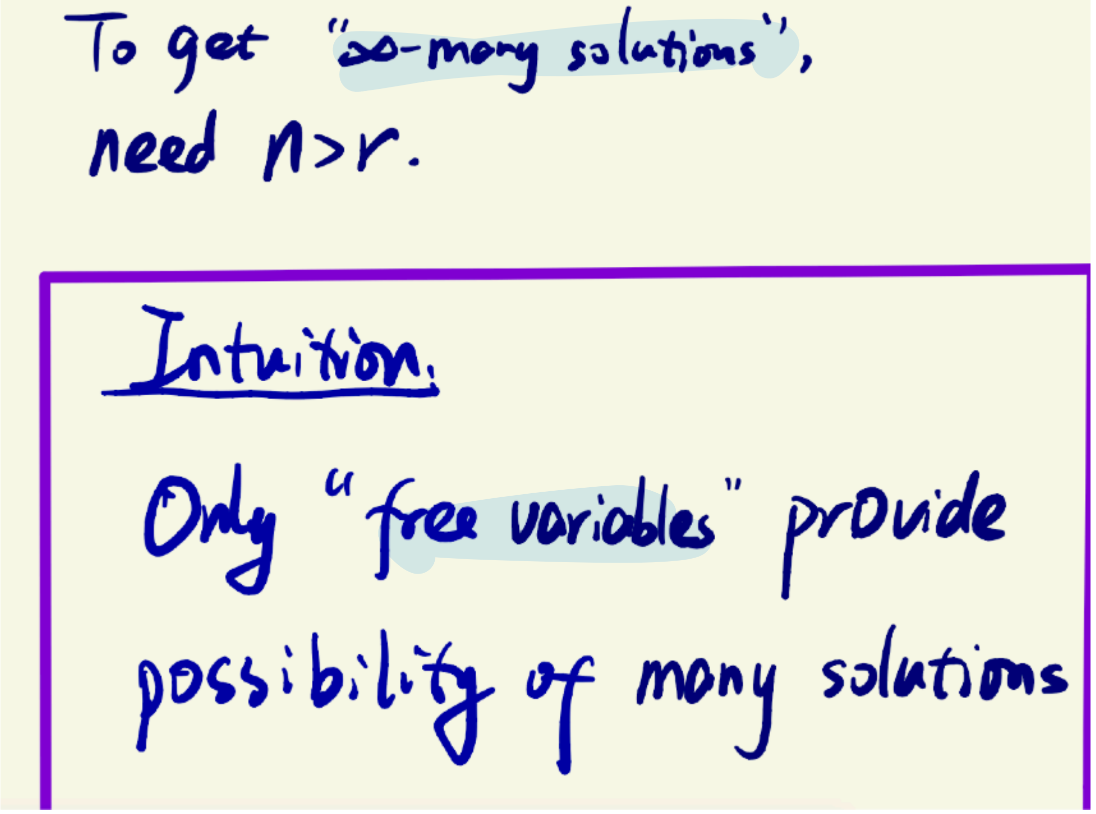
determine # of solutions



The only possibility of "No solution" is $m > r$ and $c \neq 0$.

Intuition:

Only "free rows" provide possibility of contradiction



To get "so-many solutions", need $n > r$.

Intuition:

Only "free variables" provide possibility of many solutions

Full Row rank and Full Column rank

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ | & | & | & | \end{bmatrix} \overset{\text{unique}}{\begin{bmatrix} \vdots \end{bmatrix}} = u$$

Proposition 15.1: If rank(A) = n, i.e., full column rank, then Ax = b has at most one solution (# of solutions must be 0 or 1; eliminate possibility of ∞).

This means: F (free var's) exists in RREF.

Relation: unique representation by indepedent set

$A = [v_1, \ldots, v_n]$,

where v_i's are indep.

Any $\vec{u} = \alpha_1 \vec{v_1} + \cdots + \alpha_n \vec{v_n}$ has at most one solution $\{\alpha_i\}$'s.

$\Rightarrow$ Any $\vec{u}$ can be written as $\sum_{i=1}^{n} \alpha_i \vec{v_i}$ in at most one way.

[Last time: if span$\{v_i\text{'s}\}$ = whole space, then $\vec{u}$ = unique rep of $\{v_i\text{'s}\}$

(coordinates)

Full Row rank and Full Column rank

Proposition 15.1: If rank(A) = n, i.e., full column rank, then Ax = b has at most one solution (# of solutions must be 0 or 1; eliminate possibility of ∞).

This means: F (free var's) exists in RREF.

0 or 1

$\cancel{\infty}$

Proposition 15.2: If rank(A) = m, i.e., full row rank, then Ax = b has at least one solution (# of solutions must be 1 or ∞; eliminate possibility of 0)

This means: no zero rows.

1 or ∞

$\cancel{0}$

Corollary 15.1 If rank(A) = m = n, then Ax=b has exactly one solution.

Square, then invertible

$x = A^{-1} b$.

Part II   Nullity

Strang's book Sec. 3.4

Back to Question on Expressing Solution Set

The complete solution is

$$\mathbf{x}_p + N(A) = \mathbf{x}_p + \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}\}$$

We use $n - r$ vectors.

Question: Is there a "simpler" way to express N(A)?

fewer vectors

More professional:

independent

Do these $n - r$ vectors form a basis of N(A)?

Express Solution Set Using RREF [I, F; 0, 0]

Suppose RREF $\left[\begin{array}{cc|c} I & F & \hat{b} \\ 0 & 0 & \hat{c} \end{array}\right]$. (note: after column exchange)

$$\text{Correspond}\longrightarrow \quad \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_B \\ x_F \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \hat{c} \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} I \cdot x_B + F x_F = \hat{b} \\ \qquad\quad 0 = \hat{c} \end{cases}$$

matrix

$$\xrightarrow{\text{if } \hat{c}=0} \quad \overset{\text{pivot var}}{x_B} = \hat{b} - F \overset{\text{free var}}{x_F}.$$

$\overset{N(A)}{\text{LC of col's}} \begin{bmatrix} -F \\ I \end{bmatrix}$

e.g.

$$\begin{cases} x_1 = 3 - 2x_4 \\ x_2 = 4 + 3x_4 \\ x_3 = 1 - x_4 \end{cases}$$

$$\Rightarrow x = \begin{bmatrix} \hat{b} - F x_F \\ x_F \end{bmatrix} = \begin{bmatrix} \hat{b} \\ 0 \end{bmatrix} + \begin{bmatrix} -F \\ I \end{bmatrix} x_F$$ is a solution, $\forall x_F \in \mathbb{R}^{n-r}$.

$$\Rightarrow \text{solution set } \left\{ \begin{bmatrix} \hat{b} \\ 0 \end{bmatrix} + \begin{bmatrix} -F \\ I \end{bmatrix} \vec{y} \;\middle|\; \vec{y} \in \mathbb{R}^{n+r} \right\}$$

$$\text{note: } \left\{ \begin{bmatrix} -F \\ I \end{bmatrix} \vec{y} \;\middle|\; \vec{y} \in \mathbb{R}^{n-r} \right\} \underline{\underline{\text{should be}}}\; N(A)$$

$$\Big\| \text{ by def}$$

$$C\left(\begin{bmatrix} -F \\ I \end{bmatrix}\right)$$

Are Columns Linearly Independent?

We know: $N(A) = \text{span of col's of} \overbrace{\begin{bmatrix} -F \\ I_{n-r} \end{bmatrix}}^{n-r} \begin{matrix} \}r \\ \}n-r \end{matrix}$

Is this simplifiable?

Q. Are col's of $\begin{bmatrix} -F \\ I \end{bmatrix}$ independent?

A: Yes.

Proof I Write $\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} \vec{f_1} & \cdots & \vec{f_{n-r}} \\ \vec{e_1} & \cdots & \vec{e_{n-r}} \end{bmatrix}$

$$\left(\alpha_1 \begin{bmatrix} \vec{f_1} \\ \vec{e_1} \end{bmatrix} + \cdots + \alpha_{n-r} \begin{bmatrix} \vec{f_{n-r}} \\ \vec{e_{n-r}} \end{bmatrix} = 0 \right) \Rightarrow (\alpha_1 = \cdots = \alpha_{n-r} = 0)$$

$$\Updownarrow$$

$$\left\{ \begin{matrix} \alpha_1 \vec{f_1} + \cdots + \alpha_{n-r} \vec{f_{n-r}} = 0 \\ \alpha_1 \vec{e_1} + \cdots + \alpha_n \vec{e_n} = 0 \end{matrix} \right\} \Rightarrow$$

$\vec{e_i}$'s are indep. .

$\Downarrow$

By def: col's are indep.

Proof II $\begin{bmatrix} -F \\ I \end{bmatrix} \vec{y} = 0 \Rightarrow \left\{ \begin{matrix} -F\vec{y} = 0 \\ \vec{y} = 0 \end{matrix} \right. \Rightarrow \vec{y} = 0,$

So col's are indep.

Solution of Ax=0

Linear system:

Pivot columns

Pivot variables

$$\begin{bmatrix}1&0&0&-1&4\\0&1&0&3&1\\0&0&1&2&2\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

Free variables

$$\begin{bmatrix}-F\\I\end{bmatrix}=\begin{bmatrix}1&-4\\-3&-1\\-2&-2\\1&0\\0&1\end{bmatrix}$$

Solution $x=\begin{bmatrix}1&-4\\-3&-1\\-2&-2\\1&0\\0&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=s\begin{bmatrix}1\\-3\\-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-4\\-1\\-2\\0\\1\end{bmatrix}$

The two columns of M are linearly independent.

Why?

In general, $(n-r)$ columns of M are linearly independent.

Why RREF provides (n-r) indep. Columns?

$$\begin{bmatrix} -F \\ I \end{bmatrix}$$

① $N(A)$ = span of columns of $\begin{bmatrix} -F \\ I \end{bmatrix}$

② Col's of $\begin{bmatrix} -F \\ I \end{bmatrix}$ are indep

Col's of $\begin{bmatrix} -F \\ I \end{bmatrix}$ is basis of $N(A)$

$\Downarrow$ dim

dim of $N(A)$ is $n-r$.

span

+ indep

$\Downarrow$

basis

RREF

$$\begin{array}{ccc} & r & n-r \\ r & I_r & F \\ m-r & 0 & 0 \end{array}$$

extract F

$$\begin{array}{cc} & n-r \\ 2r & \begin{bmatrix} -F \\ I \end{bmatrix} \end{array}$$

indep. columns,
form basis of $N(A)$

Nullity

Definition 15.1 (Nullity)
The dimension of the null space dim(N(A)) is called the “nullity” of A.

“Nullity” is the “true size” of solution set.

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“Nullity” is the “true size” of solution set.

Previous slide shows: dim(N(A)) = n - r. Equivalently:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem 15.1 (rank-nullity theorem)
Suppose A has n columns. Then

$$\text{rank}(A) + \dim(N(A)) = n\,.$$

generalize

Corollary 15.2 If rank(A) =n, then dim(N(A)) = 0,
i.e., Ax = 0 has a trivial solution **0**.

square; $A^{-1} \exists \iff Ax=0$ has unique solution $x=0$.

Invertibility Conditions

Theorem 15.2 (Equivalent Conditions for Invertibility)

Let $A \in \mathbb{R}^{n \times n}$.

The following statements are equivalent:

1. A is invertible
2. The linear system $A\mathbf{x} = 0$ has a unique solution $\mathbf{x} = \mathbf{0}$
3. A is a product of elementary matrices
4. A has n pivots; or equivalently: rank(A) = n
5. The columns of A span $\mathbb{R}^n$.
6. The columns of A are linearly independent.
7. The columns of A form a basis.
8. dim(C(A)) = n.
9. The linear system $A\mathbf{x} = b$ is solvable for any b.
10. dim(N(A)) = 0, or N(A) = {0}.

Equation solving

Rank

Span

Linear independence

Basis

Col space Dimension

Equation solving

Null space dim

Part III Orthogonality

Strang’s book Sec. 3.4

Motivation

To better understand rank-nullity theorem, need to introduce: Orthogonality.

Motivation: Row form of Ax=0.

Row form:

$$\begin{bmatrix} \mathbf{a}_{(1)}^{\top} \\ \mathbf{a}_{(2)}^{\top} \\ \cdots \\ \mathbf{a}_{(n)}^{\top} \end{bmatrix} \mathbf{x} = 0$$

Orthogonality

Definition 15.2 (Orthogonal)
Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal if $\mathbf{u}^\top \mathbf{v} = 0$. Denote $\mathbf{u} \perp \mathbf{v}$.

$$\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\| \mathbf{x} \| \| \mathbf{y} \|} = \mathbf{u}^T\mathbf{v}$$

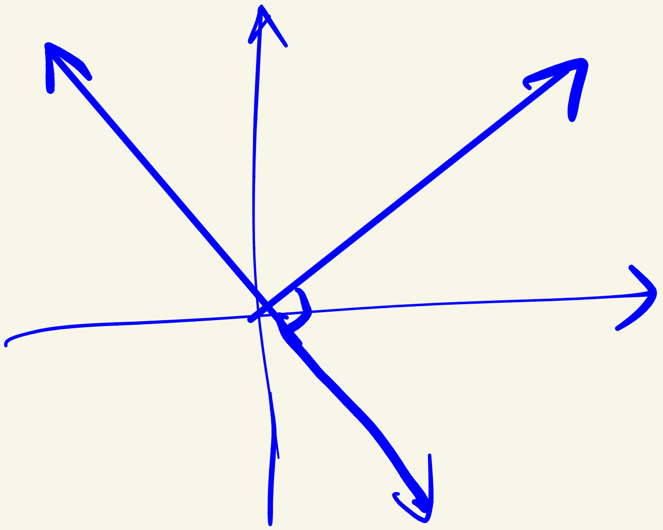
Thus
$\mathbf{x}$ and $\mathbf{y}$ are orthogonal $\Leftrightarrow \mathbf{x}^T\mathbf{y} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta$ is the right angle.

Examples:

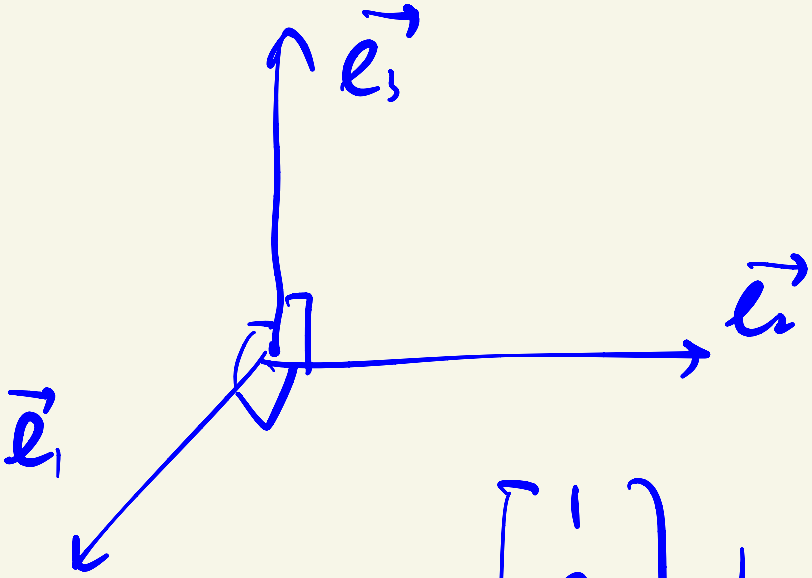
Vectors $[3, 2]^T$ and $[-4, 6]^T$ are orthogonal in $\mathbb{R}^2$.
Vectors $[2, -3, 1]^T$ and $[1, 1, 1]^T$ are orthogonal in $\mathbb{R}^3$.

Examples vector ⊥ vector.

2D



3D.



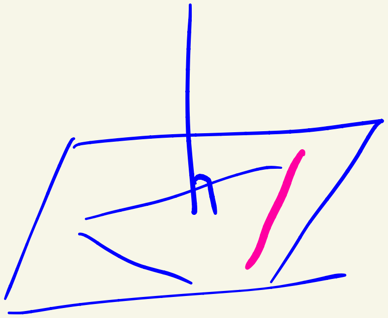
$\vec{e}_i \perp \vec{e}_j$.

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \perp \begin{bmatrix}0\\2\\3\end{bmatrix}$$

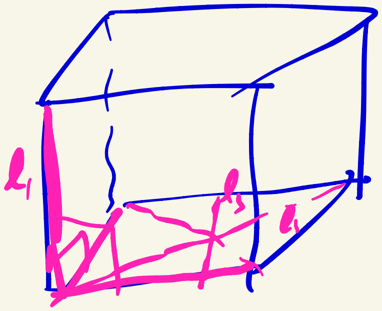
Vector ⊥ space.

High school. one line ⊥ two line,

⇒ ⊥ plane of two line.



vec ⊥ 2D space.

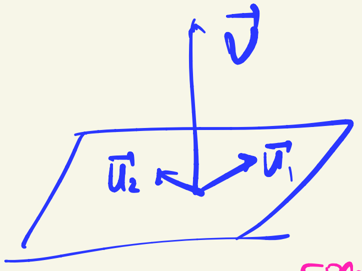


Extend, vec ⊥ any subspace U.

Def. $\vec{v} \in V$, U is a subspace of V.

$\vec{v} \perp U$ iff $\vec{v} \perp \vec{u}$, $\forall \vec{u} \in U$.

Space $\perp$ Space



$\underbrace{span(\{\vec{v}\})}_{\text{one vec}} \perp \underbrace{span(\{\vec{u}_1, \vec{u}_2\})}_{\text{two vec}}$

Can say $\overset{\text{space}}{span(\{\vec{v}_1, \vec{v}_2\})} \perp \overset{\text{space}}{span(\{\vec{u}_1, \vec{u}_2\})}$ iff $\vec{v}_i \perp \vec{u}_j$, $\forall i, j$.

Def: Subspaces U and V.
We say $U \perp V$ iff $u \perp v$, $\forall u \in U, v \in V$.

4D space

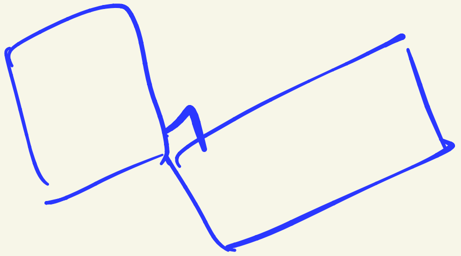


示意图 (illustration).

Orthogonal Subspaces

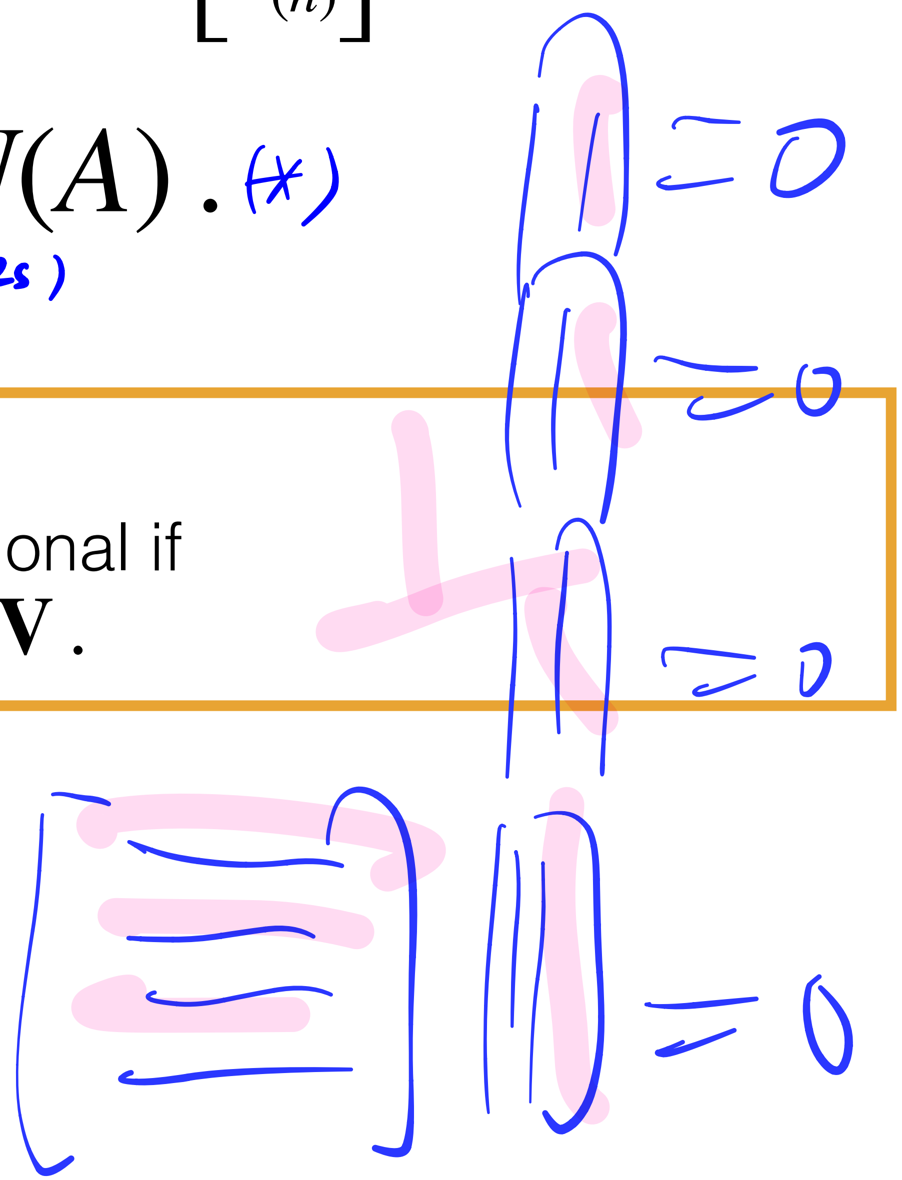
If x is in the solution set of $Ax = 0$, then $\begin{bmatrix} \mathbf{a}_{(1)}^\top \\ \mathbf{a}_{(2)}^\top \\ \cdots \\ \mathbf{a}_{(n)}^\top \end{bmatrix} \mathbf{x} = 0$

$\mathbf{a} \perp \mathbf{x}, \forall\, \mathbf{a} \in \text{Row}(A), \mathbf{x} \in N(A)\,.$ (*)

(prove in next two pages)

by Def.

Definition 15.3 (Orthogonal subspaces)
We say two subspaces $\mathbf{U}, \mathbf{V} \subseteq \mathbb{R}^n$ are orthogonal if
$\mathbf{u} \perp \mathbf{v}, \forall\, \mathbf{u} \in \mathbf{U}, \mathbf{v} \in \mathbf{V}\,.$ Denoted as $\mathbf{U} \perp \mathbf{V}\,.$



Fact: Row(A) $\perp$ N(A)

Orthogonal Subspaces

$2x_1+4x_2=0 \iff \begin{bmatrix}2\\4\end{bmatrix}\perp\begin{bmatrix}x_1\\x_2\end{bmatrix}$

If x is in the solution set of $Ax = 0$, then $\begin{bmatrix}\mathbf{a}_{(1)}^\top \\ \mathbf{a}_{(2)}^\top \\ \cdots \\ \mathbf{a}_{(n)}^\top\end{bmatrix}\mathbf{x} = 0$

$$\mathbf{x} \in N(A) \implies \mathbf{a} \perp \mathbf{x}, \forall\, \mathbf{a} \in \text{Row}(A)$$

row form

$x \in N(A) \iff Ax=0 \iff \vec{a}_{(i)}^T \vec{x} = 0,\ i=1,\dots,n.$

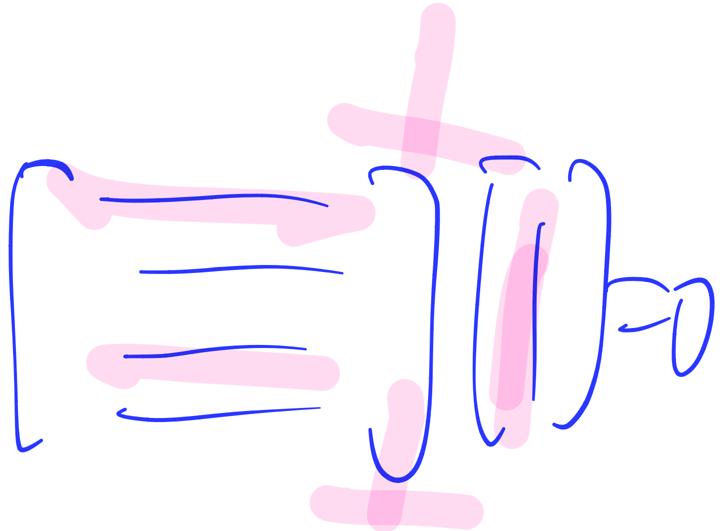
$\iff x \perp \vec{a}_{(i)},\ \forall i.$

$\iff x \perp \beta_1 \vec{a}_{(1)} + \cdots + \beta_n \vec{a}_{(n)},\ \forall \beta_1,\dots,\beta_n \in R.$

any LC of rows

$\iff x \perp \vec{a},\ \forall \vec{a} \in Row(A)$

$\iff x \perp Row(A)$



__Now__ $A\vec{x}=0 \iff \vec{x} \perp \text{Row}(A)$.

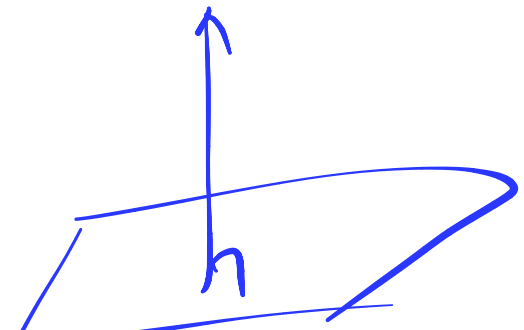
$$\underbrace{\text{any } \vec{x} \text{ satisfying } A\vec{x}=0}_{\substack{\text{all of such } \vec{x} \\ N(A)}}, \Longrightarrow \vec{x} \perp \text{Row}(A)$$

$$\Longrightarrow \quad N(A) \perp \text{Row}(A).$$

Examples and Non-example

Eg 1: Floor and vertical flag are orthogonal

vec ⊥ space



Eg 2: One line on the floor and vertical flag are orthogonal

Non-example: Floor and wall are not orthogonal.
i.e., in 3D space, xy plane and xz plane are NOT orthogonal

$\mathrm{span}(\{\vec{e}_1, \vec{e}_2\})$

$\not\perp$ $\mathrm{span}(\{\vec{e}_1, \vec{e}_3\})$

$u \in U$

$\perp$

$v \in V$,

$\vec{e}_1$

$\not\perp$

$\vec{e}_1$

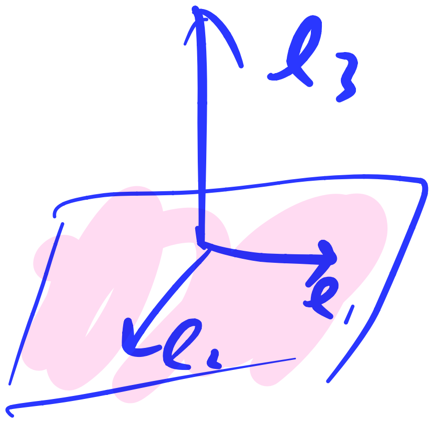
Set of All Orthogonal Vectors?

Shenzhen's all company

GDP growth. 5.4% ↗

Manager's view

In 3D space, what vectors are orthogonal to $e_3 = [0,0,1]^T$?

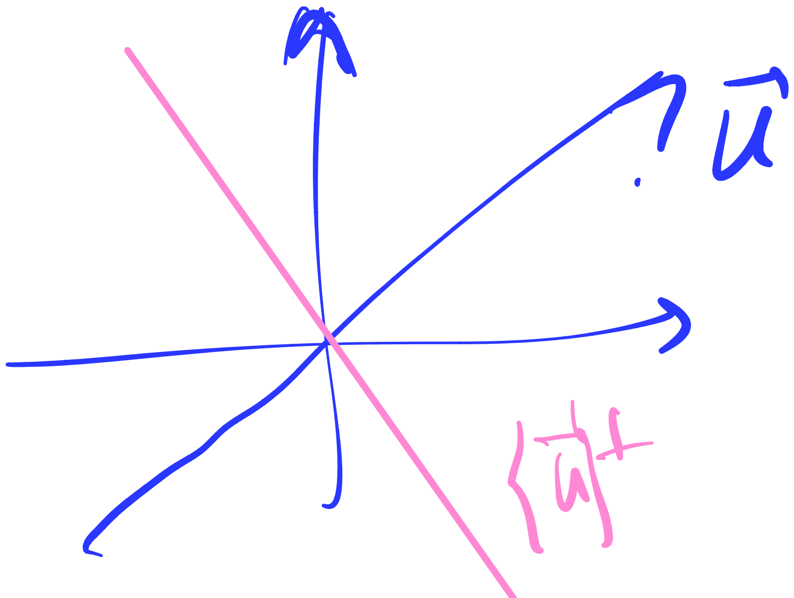


$$\begin{bmatrix}0\\0\\1\end{bmatrix} \perp \begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{e}_1, \quad \perp \begin{bmatrix}0\\1\\0\end{bmatrix} = \vec{e}_2$$

$$\perp \begin{bmatrix}1\\2\\0\end{bmatrix} = \vec{e}_1 + 2\vec{e}_2,$$

Collect all vectors that are orthogonal to $e_3 = [0,0,1]^T$, form a space?

$\{\vec{e}_3\}^\perp$ ortho complement



Orthogonal Complement of Subspace

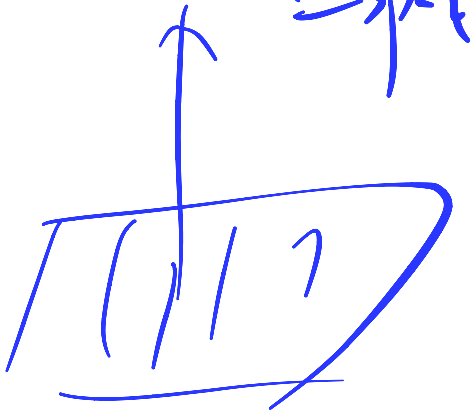
Definition: A vector v is orthogonal to subspace U iff $v \perp u, \forall u \in U$.

Diagram

There can be many vectors that are orthogonal to subspace U.
Collect all of them, we obtain span

$\mathbb{R}^3$ $\text{span}(\{\vec{e}_3\})^{\perp}$

$= \text{span}(\{\vec{e}_1, \vec{e}_2\})$



$\mathbb{R}^5$

$\text{span}(\{\vec{e}_1, \vec{e}_2\})$,

$\text{span}(\{\vec{e}_1, \vec{e}_2\})^{\perp} = \text{span}(\{e_3, e_4, e_5\})$

Orthogonal Complement of Subspace

Definition: A vector v is orthogonal to subspace U iff ________

There can be many vectors that are orthogonal to subspace U.
Collect all of them, we obtain ________

Definition 15.4 (Orthogonal complement)
For any subspace $\mathbf{V} \subseteq \mathbb{R}^n$, the set of vectors that are orthogonal to $\mathbf{V}$

$$\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u} \perp \mathbf{v}, \forall \mathbf{v} \in \mathbf{V}\}.$$

is called the orthogonal complement of $\mathbf{V}$, denoted as $\mathbf{U} = \mathbf{V}^\perp$.

(reads "V perp")

Remark: A subspace has a unique orthogonal complement!

Orthogonal Complement: Matrix Row/Null Space

Recall: Row(A) $\perp$ N(A), even stronger: $Row(A) = N(A)^{\perp}$.

If $\begin{bmatrix} \mathbf{a}_{(1)}^{\top} \\ \mathbf{a}_{(2)}^{\top} \\ \cdots \\ \mathbf{a}_{(n)}^{\top} \end{bmatrix} \mathbf{x} = 0$ then x is in the solution set of $Ax = 0$

How is this statement different from a previous slide?

N(A)

= (Solution set of $Ax = 0$)

$= \{ x \mid a_{(1)}^{\perp} x = \ldots = a_{(n)}^{\perp} x = 0 \}$

$= \{ x \mid x \perp \vec{a}_{(i)}, \forall i \}$

$= \{ x \mid x \perp \sum_{i}^{n} \beta_i a_{(i)}, \forall \beta_1, \ldots, \beta_n \in \mathbb{R} \}$

$= \{ \vec{x} \mid \vec{x} \perp \vec{a}, \forall a \in Row(A) \}$

$= Row(A)^{\perp}$

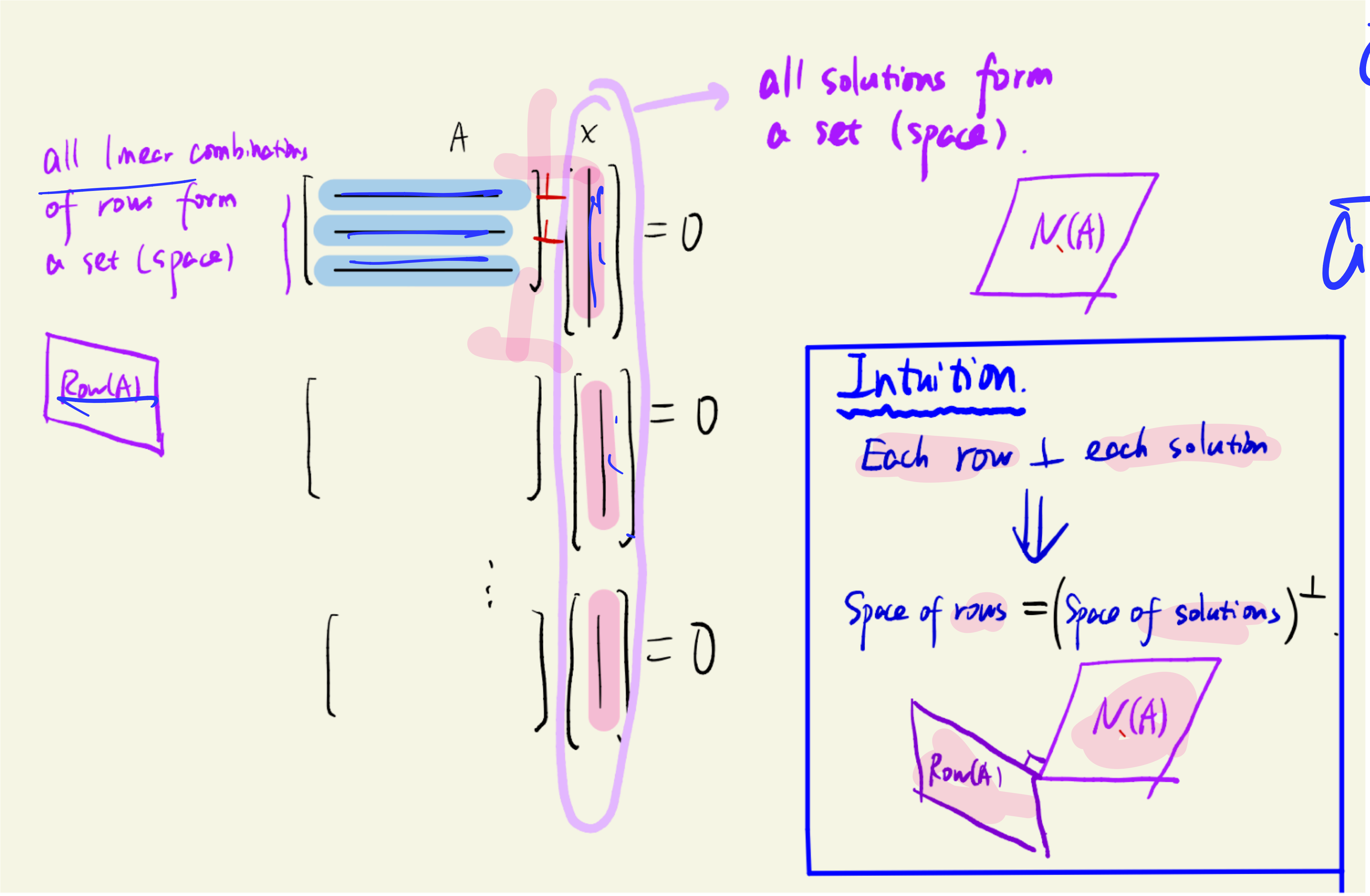
Intuition

$Ax=0$

$$\vec{a}_i^T \vec{x} = 0$$

$$\Updownarrow$$

$$\vec{a}_i \perp \vec{x}$$


all linear combinations of rows form a set (space)

all solutions form a set (space).

Intuition.

Each row ⊥ each solution

⇓

Space of rows = (Space of solutions)$^\perp$.

Examples and Non-example

Eg 1: Floor = (vertical flag)$^\perp$,
Or in 3D space, span($\{e_1, e_2\}$) = span($\{e_3\}$)$^\perp$

Eg 2: in 2D, x-axis = (y-axis)$^\perp$,
Or span($\{e_1\}$) = span($\{e_2\}$)$^\perp$

Non-example: One line on the floor $\neq$ (vertical flag)$^\perp$
Or in 3D, x-axis $\neq$ (y-axis)$^\perp$.

Facts

① $\mathrm{Row}(A) = N(A)^{\perp}$. (just now)

② $\dim(\mathrm{Row}(A)) + \dim(N(A)) = n$.

(20 mins ago)

Guess. (not just matrix)

$$\underbrace{U = V^{\perp} \subseteq \mathbb{R}^n}_{\text{combine to get whole space}} \Longrightarrow \underbrace{\dim(U) + \dim(V)}_{\text{combine dim to get}} = \underbrace{n}_{\text{whole dim}}$$

Dimension of Orthogonal Complement

Theorem 15.2 (dim of V perp)
Suppose S is a subspace of $\mathbb{R}^n$. Then $S^\perp$ is a subspace and

$$\dim(S) + \dim(S^\perp) = n\,.$$

Furthermore, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is a basis of S, and $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ is a basis of $S^\perp$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbb{R}^n$.

dim sum = n

basis combine

Proof: see next page; for reading.

→ whole spa basis

U $U^\perp$

dim → whole dim

basis → whole basis

Remark: $(S^\perp)^\perp = S\,.$

Reading: Proof of Thm 15.2

Proof.
(1) If $S = \emptyset$, then $S^{\perp} = \mathbb{R}^n$, the statement is true.
(2) Assume that $S \neq \emptyset$, then let $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ be a basis for S, let $A = [\mathbf{u}_1, \cdots, \mathbf{u}_r]$, then $S = Col(A)$, $rank(A) = rank(A^T) = r$ and

$$S^{\perp} = Col(A)^{\perp} = Null(A^T)$$

By the Rank-Nullity theorem, we have $rank(A^T) + \dim(\mathrm{Null}(A^T)) = n$, thus

$$\dim S + \dim S^{\perp} = n$$

Now suppose that the following linear combination is zero, i.e.,

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_r\mathbf{u}_r + \alpha_{r+1}\mathbf{u}_{r+1} + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

then

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_r\mathbf{u}_r = -\alpha_{r+1}\mathbf{u}_{r+1} - \cdots - \alpha_n\mathbf{u}_n$$

The LHS is a vector in S and the RHS is a vector in $S^{\perp}$, since $S \cap S^{\perp} = \{\mathbf{0}\}$, then

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_r\mathbf{u}_r = 0 = -\alpha_{r+1}\mathbf{u}_{r+1} - \cdots - \alpha_n\mathbf{u}_n$$

Since $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is a basis for S and $\{\mathbf{u}_{r+1}, \cdots, \mathbf{u}_n\}$ is a basis for $S^{\perp}$, thus

$$\alpha_1 = \cdots = \alpha_r = \alpha_{r+1} = \cdots = \alpha_n = 0$$

Thus, $\{\mathbf{u}_1, \cdots, \mathbf{u}_r, \mathbf{u}_{r+1}, \cdots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$.

Example: Revisit Solution Set of Ax=0

5 cols

Linear system:
$$\begin{bmatrix} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

pivot rows

Row(A) = $N(A)^{\perp}$
Verify Thm 15.2.

$$\text{Solution } x = \begin{bmatrix} 1 & -4 \\ -3 & -1 \\ -2 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = s \underbrace{\begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{u_4} + t \underbrace{\begin{bmatrix} -4 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{u_5}$$

(1) dim sum.

Row space dim = 3

null space dim = 2

3+2 = 5

Row space basis: $\mathbf{u}_1 = [1 \quad 0 \quad 0 \quad -1 \quad 4]^\top$
$\mathbf{u}_2 = [0 \quad 1 \quad 0 \quad 3 \quad 1]^\top$
$\mathbf{u}_3 = [0 \quad 0 \quad 1 \quad 2 \quad 2]^\top.$

(2) Basis.

① $\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}, \vec{u_5}$ indep.

② 5 vec in $\mathbb{R}^5$.

⇒ basis

Null space basis: $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} -4 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Back to Question on Expressing Solution Set

The complete solution is

$$\mathbf{x}_p + N(A) = \mathbf{x}_p + \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}\}$$

We use $n - r$ vectors.

Question: Can we use fewer vectors to express N(A)?

Back to Question on Expressing Solution Set

The complete solution is

$$\mathbf{x}_p + N(A) = \mathbf{x}_p + C(M) = \mathbf{x}_p + \alpha_1 \mathbf{v}_1 + \ldots + \mathbf{v}_{n-r} .$$

We use $n - r$ vectors.

Question: Can we use fewer vectors to express N(A)?

Answer: “no”.

Part I

Because **Rank-nullity theorem says:** For rank-r matrix A, need exactly $(n - r)$ vectors to express solution set of Ax=0.

Part II

Deeper understanding: null space is orthogonal complement of the row space, so dim is (n-r)

$$\underbrace{\begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}}_{A}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=0,$$

↓

dim of $N(A)$

$\vec{u}$ is one vec in $\mathbb{R}^5$, remaining 4 vectors $\perp \vec{u}$,

$\dim(N(A)) = 4$.

$$A: \begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 6 & 8\end{bmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=0$$

1-D row space in $\mathbb{R}^4$

$\dim(N(A)) = 3$

Part IV Four Fundamental Subspaces

Strang’s book Sec. 3.5

Matrix A

row space

colum space

null space

perp.

$N(A) = \text{Row}(A)^{\perp}$

$C(A) = \text{Row}(A^T)$

$N(A^T)$ new space

Left Null Space

Definition 15.3 (left null space)
The left null space of a matrix A is defined as $N(A^\top)$.

Think: What is this space?

$$\{\vec{y} \mid A^T\vec{y} = 0\}$$

$$\iff \{\vec{y} \mid \vec{y}^T A = 0\}$$

$$\downarrow$$

LC of row

Hint: using linear equations.

$$Ax = 0$$

$$\downarrow$$

LC of col

Left Null Space

Definition 15.3 (left null space)
The left null space of a matrix A is defined as N($A^\top$).

Think: What is this space?

Hint: using linear equations.

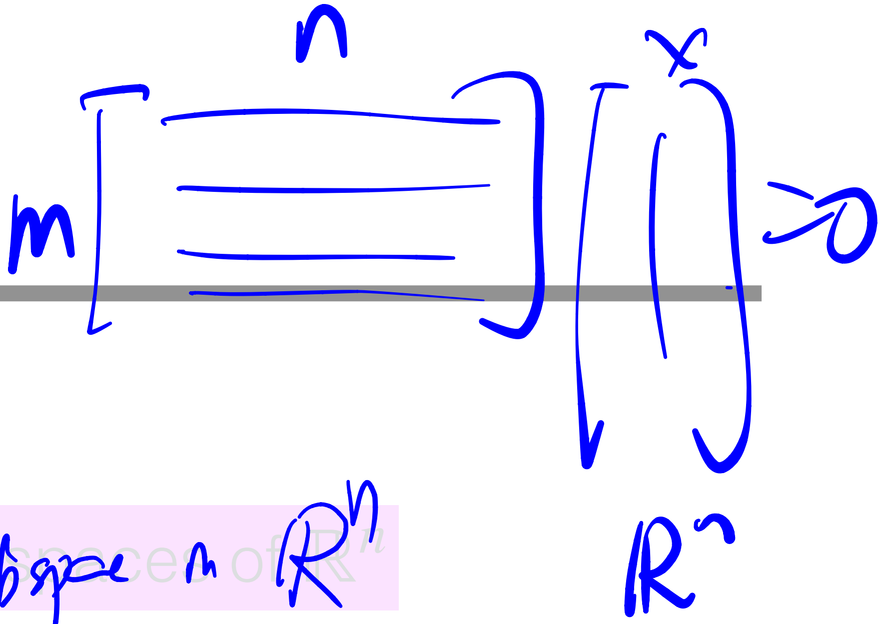
Corollary 15.2
Suppose A has m rows. Then

$$\dim(C(A)) = \boxed{\text{rank}(A)} + \boxed{\dim(N(A^\top))} = \underline{m}.$$

Row space dim + Left null space dim $= m$.

Proof: Consider $B = A^\top$

Four Fundamental Subspaces



C(A^T), subspace in $\mathbb{R}^n$

Row space dim r

ortho complement

sum = n

null space dim $n-r$

N(A), $\subseteq \mathbb{R}^n$

Four Fundamental Subspaces

C(A), Subspaces of $\mathbb{R}^m$

Column space dim r = **Row space** dim r

C(A^T), Subspaces of $\mathbb{R}^n$

ortho-comp | sum = m

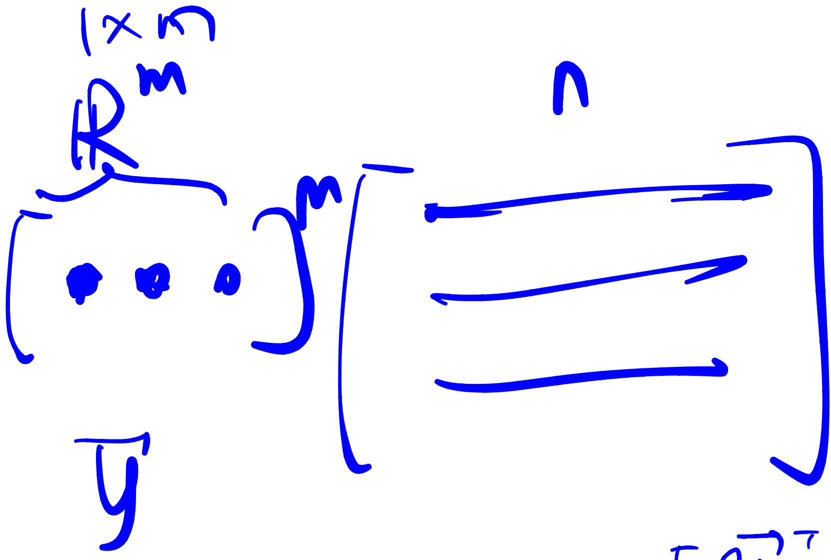
Ortho-complement | sum = n

Left null space dim $m-r$

N(A^T), Subspaces of $\mathbb{R}^m$

null space dim n-r

N(A), Subspaces of $\mathbb{R}^n$

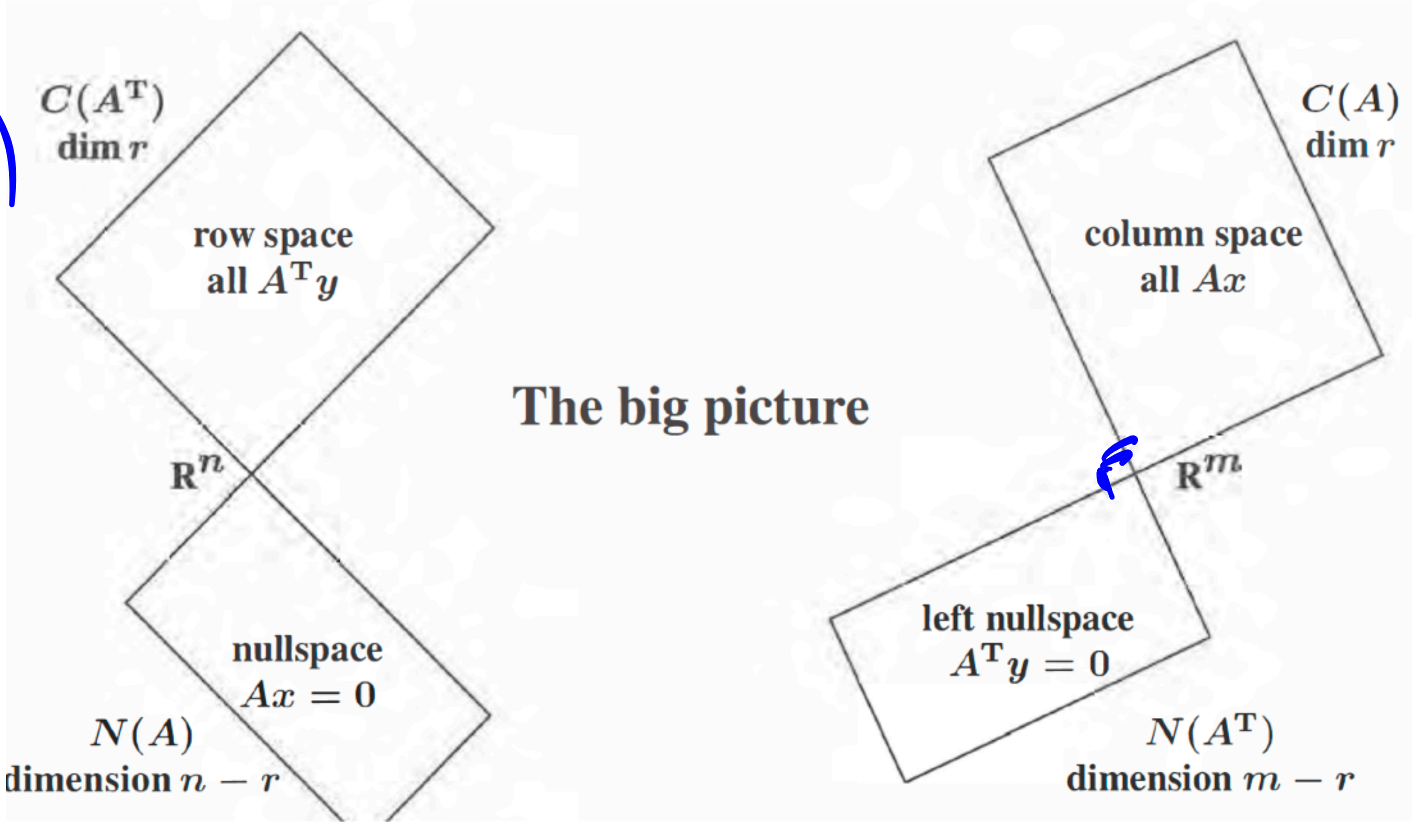


$[y_1, y_2, \cdots y_m]$ $\begin{bmatrix} \vec{v}_1^T \\ | \\ \vec{v}_m^T \end{bmatrix}$

Fundamental Theorem of Linear Algebra

Theorem 15.3 (Fundamental Theorem of Linear Algebra)
Suppose $A \in \mathbb{R}^{m \times n}$. Then:
(1) $N(A) = \text{Row}(A)^{\perp}, \quad \dim(N(A)) = n - \text{rank}(A)$.
(2) $N(A^{\top}) = C(A)^{\perp}, \quad \dim(N(A^{\top})) = m - \text{rank}(A)$.

Row(A)



$\mathbb{R}^n$

$\mathbb{R}^m$.

Summary of Solving Rectangular Linear Systems

What have we learned, for solving Ax=b?

1) For any linear system, can transform to RREF. Lec 12

2) From RREF, can write the solution set as $X_p + N(A)$ Lec 13

$\downarrow$

$(n-r)$ vec

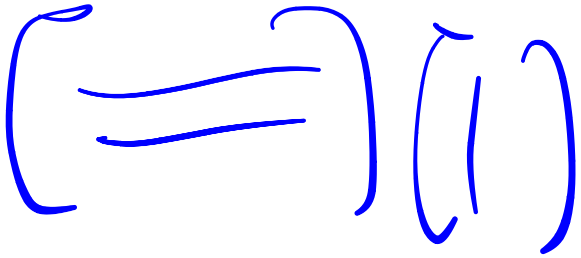
where N(A) = span of $(n-r)$ vectors

3) The solution set Cannot be simplified (use $< (n-r)$ vec) Lec 14-15

rank-nullity theorem

A new view for solving Ax=0: Lec 15

It is just finding ortho complement of Row(A)



Summary Today (write Your Own)

One sentence summary:

Detailed summary:

Summary Today (of Instructor)

One sentence summary:

Four Fundamental Subspaces

Detailed summary:

1. Full rank

—Full row rank and full column rank ==> # of solutions

2. Nullity

—Nullity = dimension of null space.

—Rank-Nullity theorem: Nullity + rank = n.

3. Orthogonality

—Orthogonal complement $V^{\perp}$ has dimension $n - dim(V)$

4. Four fundamental subspaces

—null space = (row space)$^{\perp}$; left null space = (Column space)$^{\perp}$.

—dim: $m - r, n - r$.

Mid-term exam:
Time: Nov 5, Sunday, 16:30-18:30.
Place: Stadium.

Range: Lec 1-14.