Lecture 18

Determinant

行列式

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Roadmap of Semester

Segment 1 Solving linear systems (Lec 3-15);

Segment 2

Next [Lec 16-20]: Three relatively independent parts:
—**Orthogonality**. Least squares; orthogonal matrix (Lect 16,17)
—**Determinant**. [Important tool!] Lec 18
—**Linear transformation**. [Advanced math perspective of matrix]

They are not directly related to solving system/problem,
but are fundamental and useful tools.

Segment 3 **Lec 21-26: Eigenvalues and singular values.**

Today's Lecture: Outline

Main topic: Determinant

1. Motivation: Condition for Invertibility

2. Definition of Determinants by Geometry and Derivation

3. Properties of Determinants

Strang's book: Sec 5

Other classes:

(Motivation)

↓

Definitions

↓

Properties

This class:

Motivation

↓

Desired Properties

↓

Definition

Today’s Lecture: Learning Goals

After the lecture, you should be able to

1. Tell the geometrical meaning of determinant
2. Use the properties to compute determinant of a matrix

Advanced goal:

Learn how to compute a complicated quantity based on simple properties. [“Simplification Framework"]

Review

Recall: Equivalent Conditions for Invertibility

Theorem 15.2+ (Equivalent Conditions for Invertibility)

Let $A \in \mathbb{R}^{n \times n}$

The following statements are equivalent:

1. A is **invertible**
2. The linear system $A\mathbf{x} = 0$ has a unique solution $\mathbf{x} = \mathbf{0}$
3. A is a product of elementary matrices
4. A has n pivots; or equivalently: $\text{rank}(A) = n$
5. The columns of A span $\mathbb{R}^n$ → L.S
6. The columns of A are linearly independent → L.I
7. The columns of A form a basis of $\mathbb{R}^n$
8. $\dim(C(A)) = n$
9. $\dim(N(A)) = 0$ or $N(A) = \{\mathbf{0}\}$

→ not computable directly

Part I Motivation

Recall: Homework Problems

Homework problem: When is a 2×2 matrix invertible?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$ad - bc \neq 0$

Recall: Homework Problems

Homework problem: When is a 2×2 matrix invertible?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Answer: Iff $ad - bc \neq 0.$

Advantage over the 9 conditions in Thm 15.2:

Closed-form expression to judge invertibility

Remark: We had an expression of inverse, but:

—no expression of “judging invertibility”;

—rely on an algorithm.

Motivation: Find an Invertibility Condition

Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix

行列式

Question 18.1: [extension of invertibility condition]

"to help determine invertibility" (other things)

Can we extend $(ad - bc)$ to $n \times n$ matrix, i.e. find a **determinant** **condition** $\det(\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}$ (actually a function) such that

$$\det(A) = 0 \text{ iff } A \text{ is invertible?}$$

Question 18.2: WHAT is $ad - bc$?

The fundamental understanding can help answer Q1.

How to Tackle Q2 at all?

Open-ended questions are often hard.

But there are patterns.

Powerful view to tackle such “what is” question:

GEOMETRY!

Understanding $ad - bc$

Homework problem: When is a 2×2 matrix invertible?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Answer: Iff $ad - bc \neq 0$

ad relation

$\left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} \right)$

Question 18.2: WHAT is $ad - bc$?

Understand by GEOMETRY! (By Cauchy)

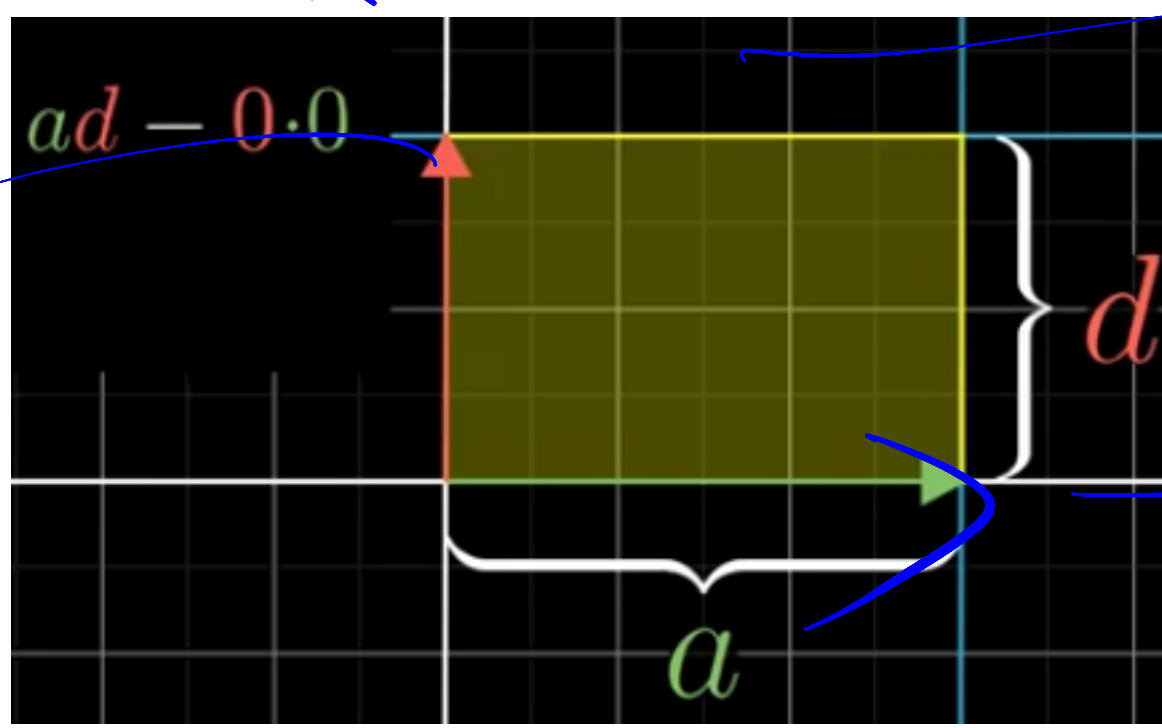
Special case: $b = c = 0$

$ad - bc = a \cdot d$

Area

面积

$\begin{bmatrix} d \\ 0 \end{bmatrix}$



$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$

Understanding $ad - bc$

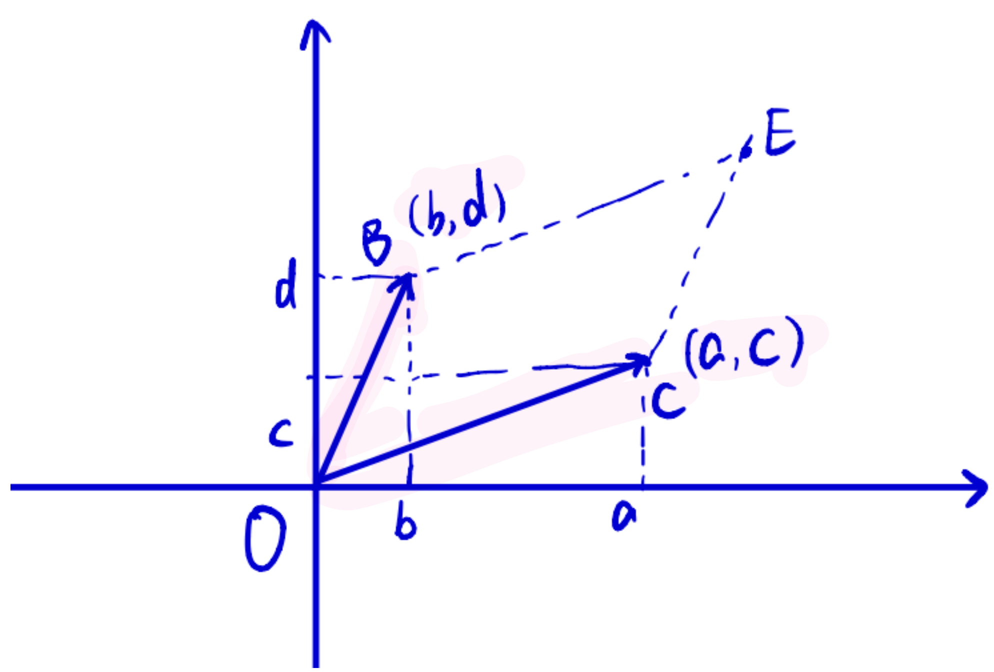
Homework problem: When is a 2×2 matrix invertible?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Answer: Iff $ad - bc \neq 0.$

Question 18.2: WHAT is $ad - bc$?

Understand by GEOMETRY! (By Cauchy)



Last page: rectangular. (if $b=c=0$).

For general a, b, c, d, column vectors $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ form parallelogram OCEB

Exercise Try to find the relation of $ad - bc$ and parallelogram OCEB

$$ad-bc \xrightarrow{\text{relation}} \mathrm{PARA}\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right).$$

Understanding $ad - bc$

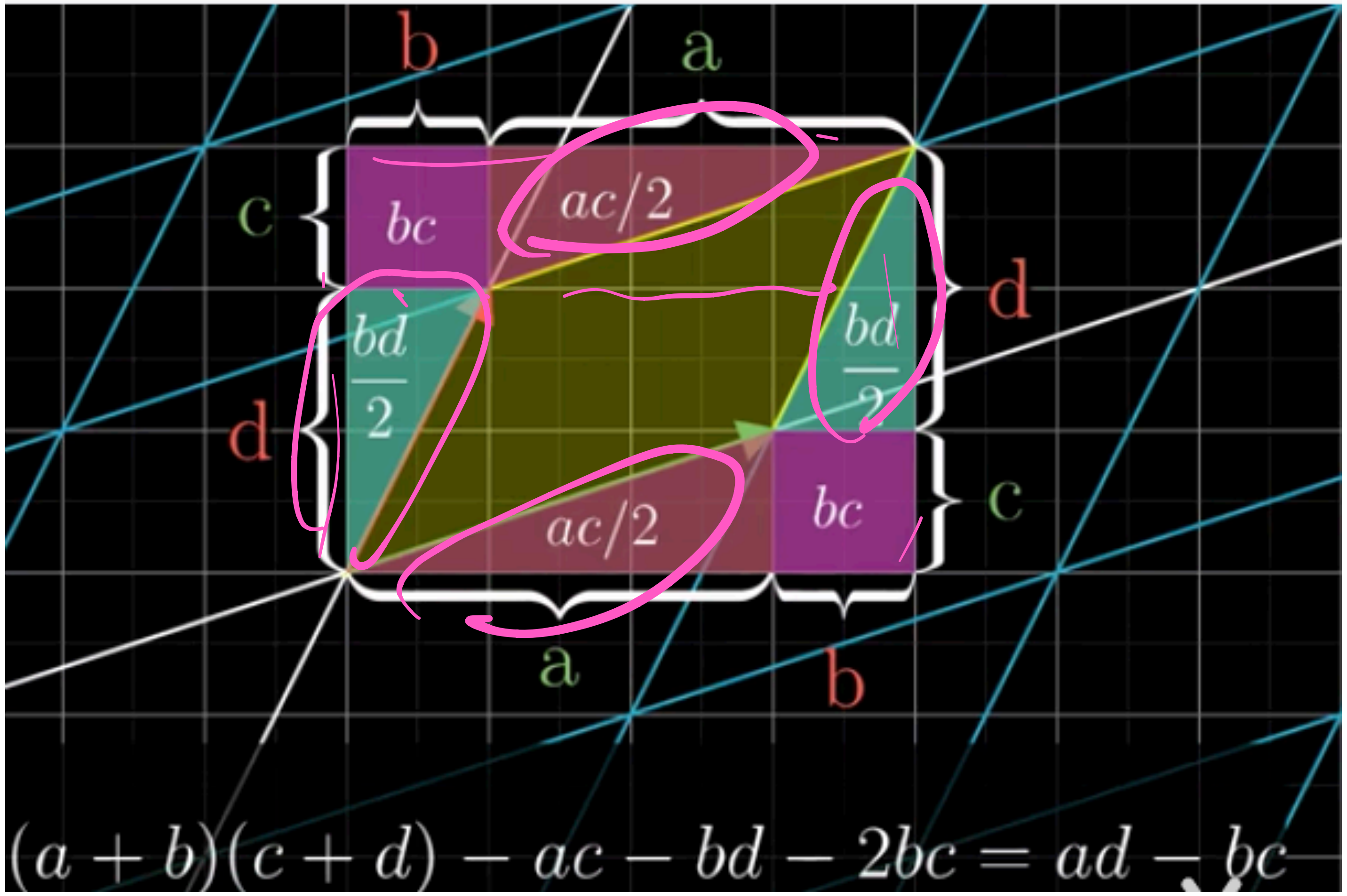
Homework problem: When is a 2×2 matrix invertible?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Answer: Iff $ad - bc \neq 0$

Question 18.2: WHAT is $ad - bc$?

Understand by GEOMETRY! (By Cauchy)



Answer: The area of a parallelogram formed by the column vectors!

Claim

$ad - bc$ = area of parallelogram formed by columns $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$.

= (area of big rectangulor)

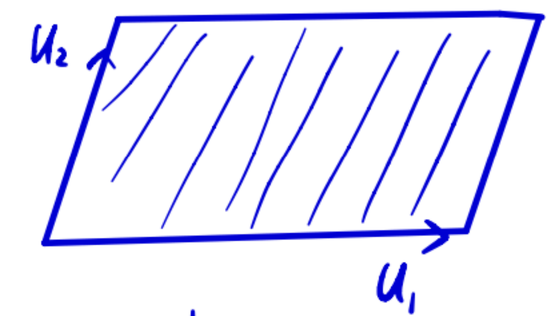
− (area of small parts)

= $ad - bc$.

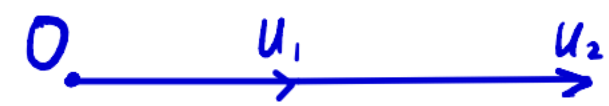
Degenerate $ad - bc$

Question 18.2: WHAT is $ad - bc$?

Area of the parallelogram
(formed by two column vectors from the matrix)!



degenerate



What does $ad - bc$ = 0 mean?

It means the area of the parallelogram is 0;

(same line)

Equivalently, the two column vectors are parallel;

dependent,

Equivalently, the matrix is NOT invertible

Understanding $ad - bc$

Question 18.2: WHAT is $ad - bc$? Area of the parallelogram

When is a 2×2 matrix invertible? Iff $ad - bc \neq 0.$

New Interpretation: A is invertible iff the area of the parallelogram is non-zero.

Logic chain (逻辑链): (geometry)

We observe: $ad - bc$ **is the area of the parallelogram of columns.**

$ad - bc$ **= 0 ⇔ the area of the parallelogram of columns = 0.**

⇔ columns on the same line

⇔ columns dependent

⇔ A not invertible.

We knew (from homework; algebraic proof)

$ad - bc = 0 \iff A$ not invertible.

Extension to Higher Dimension

Question 18.1: [extension of invertibility condition]

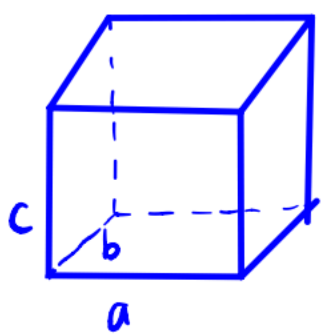
Can we extend $(ad - bc)$ to $n \times n$ matrix, i.e. find function det() s.t. $\det(A) = 0$ iff A is invertible?

Answer: Let's extend the geometrical meaning of "area" to higher dimension!

$\downarrow$ extend

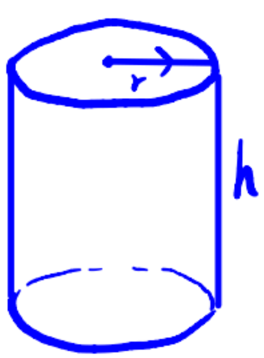
Volumn (体积)

Examples of computing volume:



Cuboid [长方体]

$V = abc$



Cylinder [圆柱体]

$V = h \cdot (\text{area of disk})$
$= h \cdot \pi r^2$

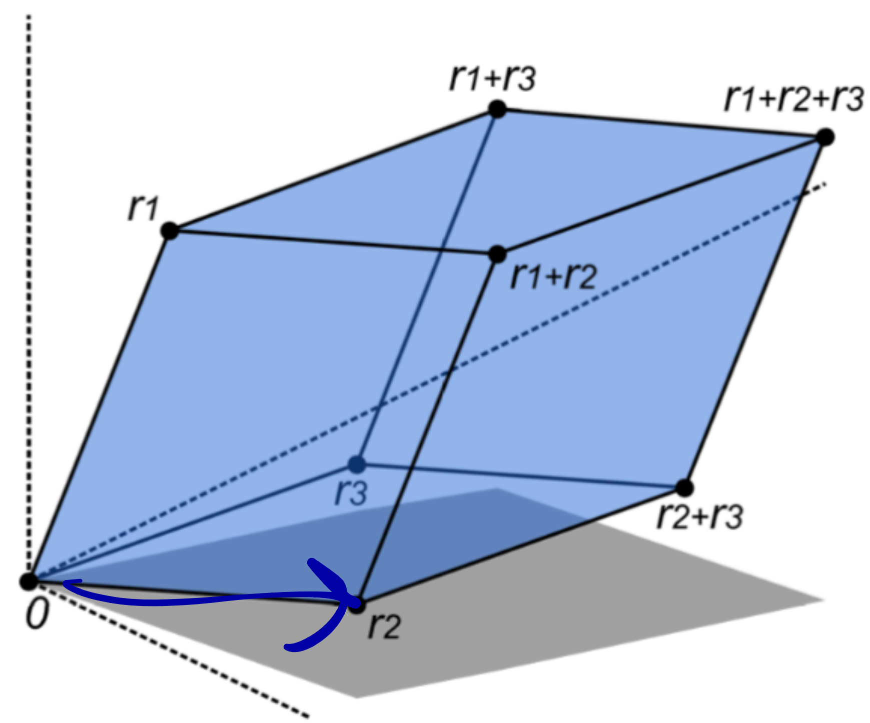
Extension to Higher Dimension

Question 18.1: [extension of invertibility condition]

Can we extend $(ad - bc)$ to $n \times n$ matrix, i.e. find function det() s.t.

$\det(A) \neq 0$ iff A is invertible?

Answer: Let's extend the geometrical meaning of "area" to higher dimension!



"Volume"!

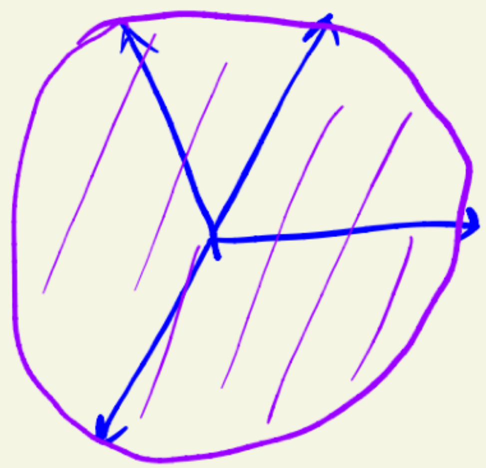
Volume of polytope formed by $\vec{r_1}, \vec{r_2}, \vec{r_3}$,

其 $A = [\vec{r_1}, \vec{r_2}, \vec{r_3}] \in R^{3\times3}$

Higher Dimension

Higher dimension (dim = 4, 5, 100?)

http://HighDimsetformedbyvectorsHigherd.me



n column vectors of $A \in \mathbb{R}^n$ form a set in $\mathbb{R}^n$.

"Volumn" is the indicator of the size of the set, i.e.. measuring how large the set is.

Remark. [for reading] The set is $\left\{ \sum_{i=1}^{n} \alpha_i \vec{v}_i : 0 \le \alpha_i \le 1, i=1,\cdots,n \right\}$.

It's a polytope.

高维多面体

Part II   Properties and Expression of Determinant (1) Decomposition

Extension to Higher Dimension

Question 18.1: [extension of invertibility condition]

Can we extend $(ad - bc)$ to $n \times n$ matrix, i.e. find function det() s.t.

$\det(A) = 0$ iff A is invertible?

"Volume"!

How to compute the "volume"? (of polytope)

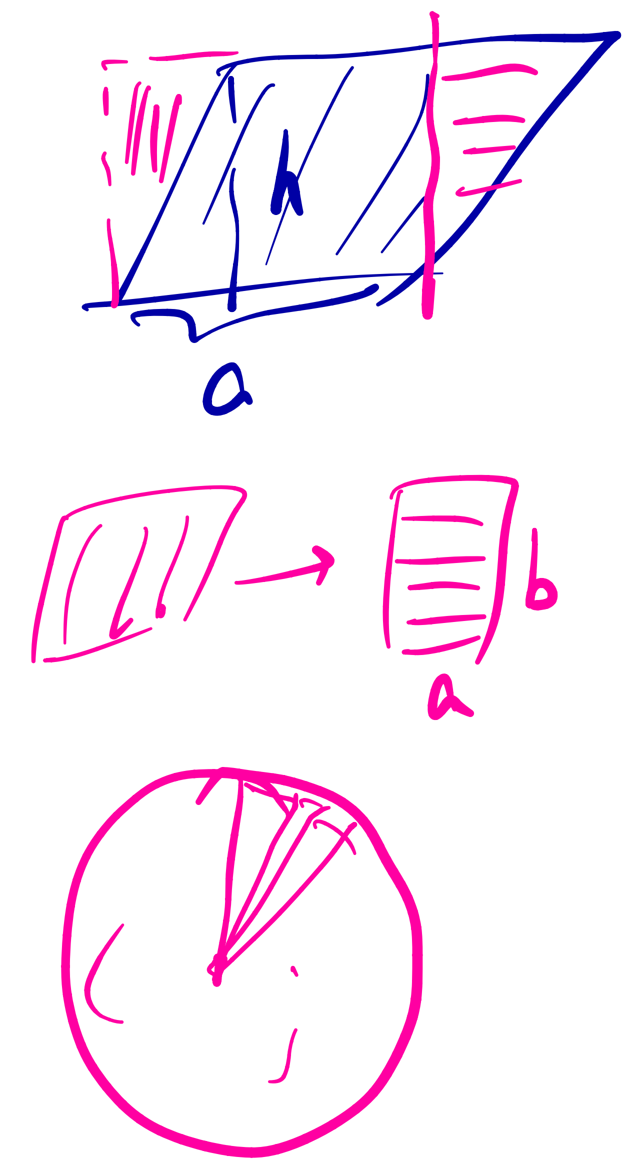
Philosophy: If you find it hard to think about the general case, then it's helpful to start with the simplest special case...

Recall: How to Compute Area from Beginning?

→ How did we calculate area of ?

1) Simplify: Relate to

2) Basic "area": Know areas of



An Original-Unit Framework

How to compute "volume"?
Idea: **Simplify**!

An Original-Unit Framework [or you name it]

(Req1) Relate A to simpler matrices B_i.　简化

(Req2) Building block: Compute the "volume" of simpler matrices　基本单元

Is this enough?

An Simplification Framework

How to compute "volume"?
Idea: **Simplify**!

An Simplification Framework [or you name it]

(Req1) Relate A to simpler matrices B_i . 简化

(Req2) Building block: Compute the "volume" of simpler matrices 基本单元

Is this enough?

(Req3) **Original-Unit Property**: Relation of det(A) and det(B_i). 原始-单元关系

Critical !!

Next 1 hour

"Volume" has "properties";

→ Extract these properties;

→ Use properties to compute determinant

2x2 determinants.

- Row operation of 2x2 matrix

Step 1 2x2

- Multiply by α: $\begin{bmatrix} a\cdot\alpha & b\cdot\alpha \\ c & d \end{bmatrix} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$a\alpha\cdot d - b\alpha\cdot c = \alpha(ad-bc)$$

Step 2 Geometry: Provide a geometrical interpretation; Geometrical properties can generalize. This leads to a generalization of 2x2 property.

Step 3 Desired property of $n\times n$ matrix.

(Req3) Relating Original and Units

(Req3) **Original-Unit Property**: Relation of det(A) and det(B_i). 原始-单元关系

Notation: Denote $|A|$ = det(A)

Desired Property P1:
The determinant is a linear function of each row separately.

When $n = 2$:

(P1.1) **multiply row 1 by any number *t***

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(P1.2) **Add row 1 of *A* to row 1 of *B*:**

$$\begin{vmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}$$

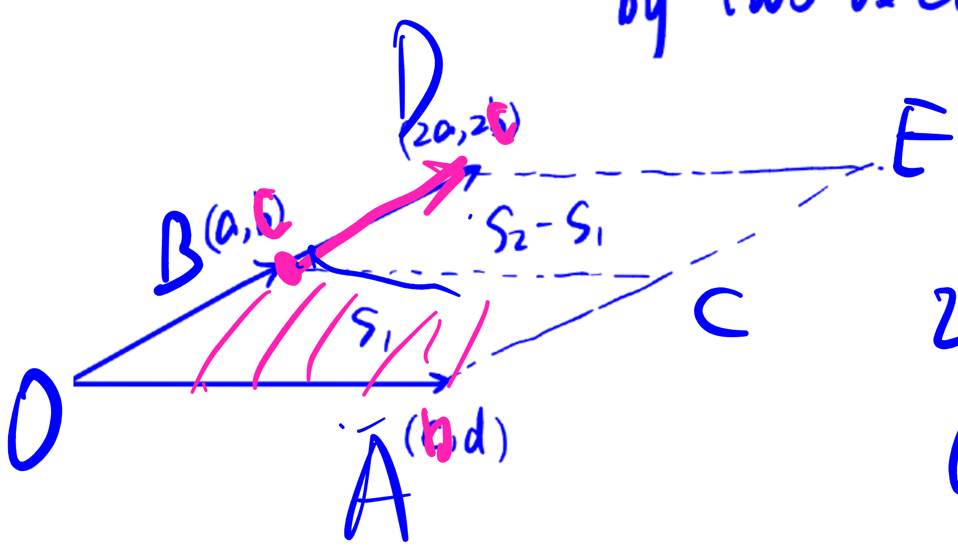
P1.1 Scaling a Row

multiply row 1 by any number *t*

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Step 2: Verify:

Geometrically: Denote PRL[(a,b), (c,d)] as the parallelogram formed by two vectors (a, b), (c,d).



What's the relation of the areas of

2·then

OACB = PARA$\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right)$

and OAED = PARA$\left(\begin{bmatrix} 2a \\ 2c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right)$

Step 3: In high-dim, volumn doubles if one side doubles.

Algebraic property: $\begin{vmatrix} \vec{a}_1 \cdot t \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{vmatrix} = t \begin{vmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{vmatrix}$

Exercise (row-wise scaling).

What is $\begin{vmatrix} 2a & 2b \\ 2c & 2d \end{vmatrix}$?

$= 2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. T or F.

$= 2\begin{vmatrix} a & b \\ 2c & 2d \end{vmatrix} = 4\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Sum of rows (just one row).

Step 1 Algebra for 2x2 case.

$$\begin{vmatrix} a_1+a_2 & b_1+b_2 \\ c & d \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}.$$

$$\begin{vmatrix} a_1+a_2 & b_1+b_2 \\ c & d \end{vmatrix} = (a_1+a_2)d - (b_1+b_2)c = (a_1 d - b_1 c) + (a_2 d - b_2 c) = \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}$$

Remark

$$\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c & d \end{bmatrix} \neq \begin{bmatrix} a_1 & b_1 \\ c & d \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c & d \end{bmatrix}.$$

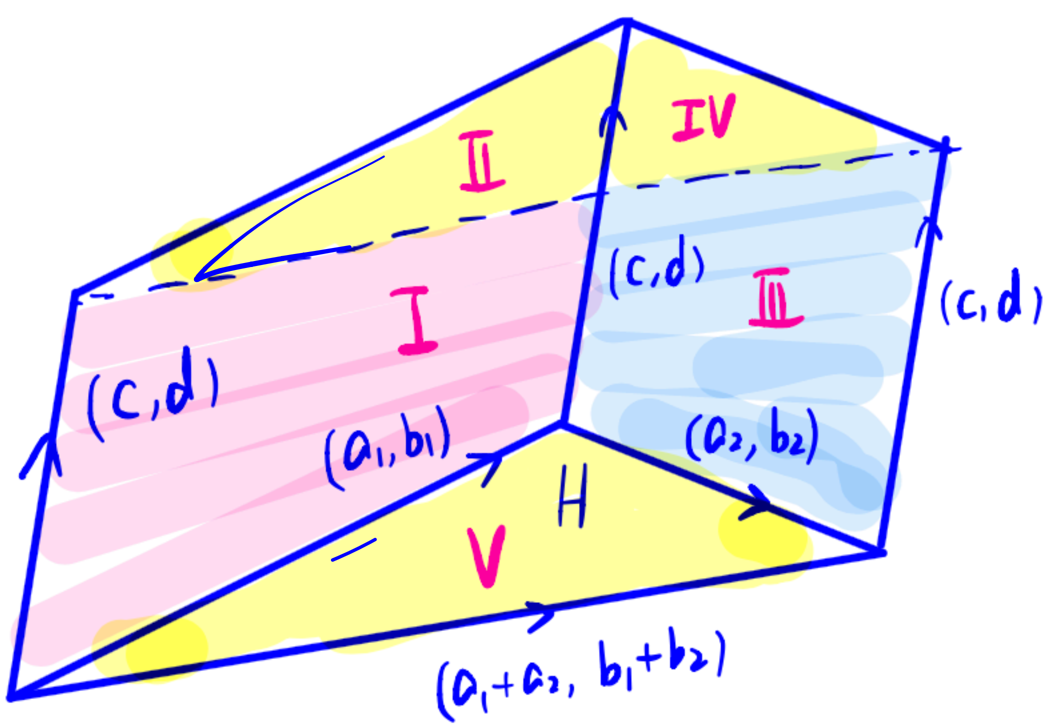
P1.2 Adding Single Row

Step1 algebraic

Add row 1 of *A* to row 1 of *B*:

$$\underbrace{\begin{vmatrix} a_1+a_2 & b_1+b_2 \\ c & d \end{vmatrix}}_{S_0} = \underbrace{\begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix}}_{S_1} + \underbrace{\begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}}_{S_2}$$

Verify: $\underbrace{\text{Area of PAR}[(a_1+a_2,\ b_1+b_2),(c,d)]}_{S_0} = \underbrace{\text{Area of PAR}[(a_1,b_1),(c,d)]}_{S_1}$
$+ \underbrace{\text{Area of PAR}[(a_2,b_2),(c,d)]}_{S_2}$.



$$\left.\begin{cases} S_0 = \mathrm{I} + \mathrm{III} + \mathrm{V} \\ S_1 = \mathrm{I} + \mathrm{II} \\ S_2 = \mathrm{III} + \mathrm{IV} \end{cases}\right\} \Rightarrow S_0 = S_1 + S_2$$

$$\mathrm{V} = \mathrm{II} + \mathrm{IV}$$

(yellow).

High-dim: This relation should be true.

$$\det\begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{bmatrix} + \det\begin{bmatrix} \vec{b_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{bmatrix} = \det\begin{pmatrix} \vec{a_1}+\vec{b_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{pmatrix}$$

Sum instead of Single-row-sum

det(A+B) = det(A) + det(B)? T or F ?

$$\begin{vmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix}$$

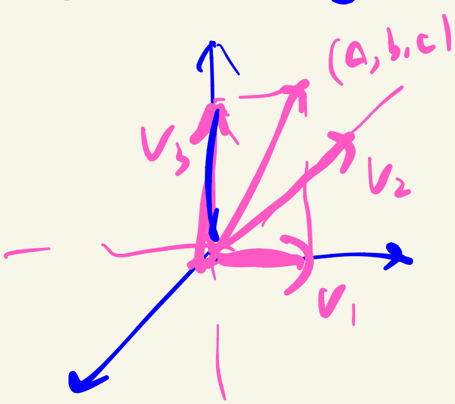
$\parallel$ 4 term

2 terms

Row Decomposition

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix} + \det\begin{bmatrix} 0 & b & 0 \\ x & x & x \\ x & x & x \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & c \\ x & x & x \\ x & x & x \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \end{bmatrix}$$



Applying to n=3

(Req1) Relate A to simpler matrices B_i. 简化

$$A = \text{Row-sum}(B_1, B_2, B_3)$$

row decomposition relation (NOT $A = B_1 + B_2 + B_3$)

(Req3) **Original-Unit Property**: Relation of det(A) and det(B_i). 原始-单元关系

$$|A| = |B_1| + |B_2| + |B_3|$$

Together:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Part III   Properties and Expression of Determinant (2)
Size Reduction

Idea

$$\det\begin{bmatrix} a_{11} & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} = \det\left[\begin{array}{c|cc} a_{11} & 0 & 0 \\ \hline 0 & \times & \times \\ 0 & \times & \times \end{array}\right] + \det\begin{bmatrix} 0 & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

$$\left[\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & B \end{array}\right]$$

"block" diagonal

→ special case of $\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$

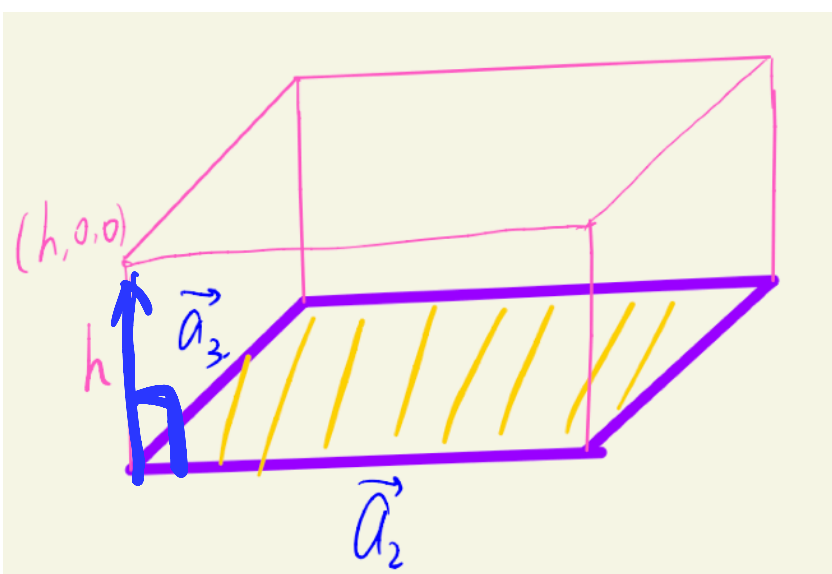
all-zero column,

Tool 2: Determinant of Block Matrix

<u>Desired Property P3</u> Suppose $\underbrace{A}_{n\times n} = \begin{bmatrix} \overset{1\times 1}{h} & \overset{1\times(n-1)}{0} \\ \underset{(n-1)\times 1}{0} & \underset{(n-1)\times(n-1)}{B} \end{bmatrix}$. Then $\det(A) = h \cdot \det(B)$.

We verify it for $n=3$, i.e., $A = \begin{bmatrix} h & 0 & 0 \\ 0 & \boxed{a_{22}} & \boxed{a_{23}} \\ 0 & \boxed{a_{32}} & \boxed{a_{33}} \end{bmatrix}$, with columns $\vec{a}_2$, $\vec{a}_3$

$$\begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \perp \text{span}\left(\begin{bmatrix} 0 \\ x \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ x \end{bmatrix} \right)$$



Volumn of Poly$\left(\begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a_{22} \\ a_{32} \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ x \end{bmatrix} \right)$

= height × bottom area

$= h \cdot \text{area}(\text{PARA}(\vec{a}_2, \vec{a}_3))$

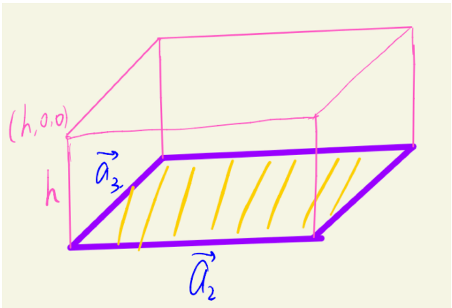
$= h \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

"parallogram" + "vertical line"

底面积 × "高" (bottom area × height)

Corollary: All-Zero Rows or Columns

Corollary: If one row or column of a square matrix A is zero, then det(A) = 0.



Collapse

Volumn of parallogram (2D)

= 0

(= height × area = 0 × area = 0).

Alternative Proof: Using 0 = -0 = (-1) · 0

$$\underbrace{\det\begin{bmatrix}\vec{0} & \vec{a}_2\end{bmatrix}}_{K} = \det\begin{bmatrix}(-1)\cdot\vec{0} & \vec{a}_2\end{bmatrix} \overset{\text{Prop 1.1}}{=} \underbrace{-1\cdot\det\begin{bmatrix}\vec{0} & \vec{a}_2\end{bmatrix}}_{-K}$$

$\Rightarrow K = 0.$

Reducing n*n to (n-1)*(n-1) Matrix

Applying Desired Property 1.2 (along 1st column), we get

$$\begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & p & q \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & p & q \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & 0 & 0 \\ d & e & f \\ g & p & q \end{vmatrix}}_{=0}$$

use http://wegetlgyaodefgpg-HiITObyGro20.lu $\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ g \end{bmatrix}$

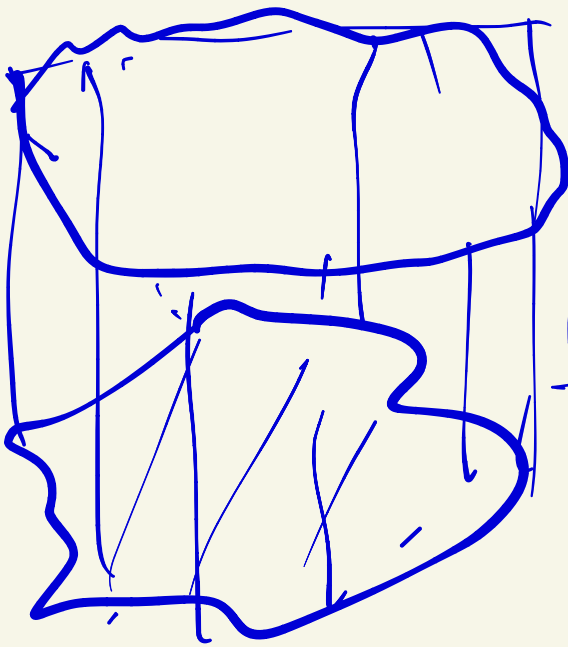
$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ g \end{bmatrix} \qquad = a \cdot \begin{vmatrix} e & f \\ p & q \end{vmatrix}.$$

Reduce to 2x2 matrix det.

In general: reduce $(n \times n)$ to $(n-1) \times (n-1)$.

Extension of high school

Volume formula



height × Area

3D volume

relation

2D area

Computing Each Term: Reduce Size

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

1st term

(Req2) Building block: Compute the "volume" of simpler matrices 基本单元

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad ①$$

same shape of $\begin{bmatrix} \times & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \overset{?}{=} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad ②$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \overset{?}{=} \begin{vmatrix} a_{12} & 0 & 0 \\ a_{22} & \times & \times \\ a_{32} & \times & \times \end{vmatrix} \quad ③$$

operation

Operation: column exchange

Desired Property 3: Swapping Columns

Desired Property P3:
Swapping columns —-> changes the sign of the determinant.

Check n =2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -\begin{vmatrix} b & a \\ d & c \end{vmatrix} = -(bc - ad)$.

relation

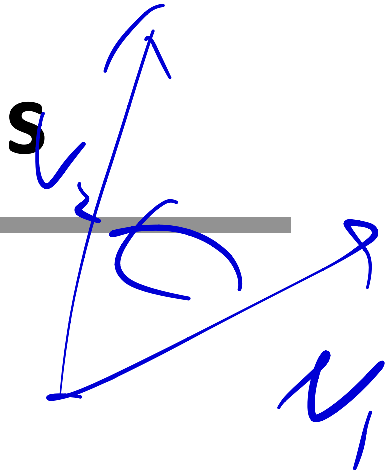
negative

Not that we enforce P3, but P3 needs to be this form.
Otherwise, contradict definition of n=2.

Apply to n=3:

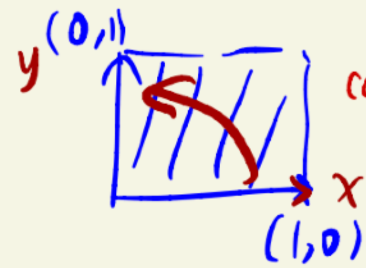
$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} = \det \begin{bmatrix} a_{12} & 0 & 0 \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Geometry Interpretation of Swapping Columns



$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ $\longrightarrow$ "area"

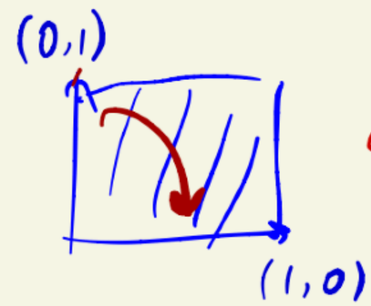
E.g. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$



counter-clockwise: "+"

$\det[e_1, e_2] = 1$ $+$

E.g. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$



clockwise "+"

$\det[e_2, e_1] = -1,$ $-$

order matters.

Two determinants are different;
Both are rectangulars formed by e_1, e_2.
How to differentiate?

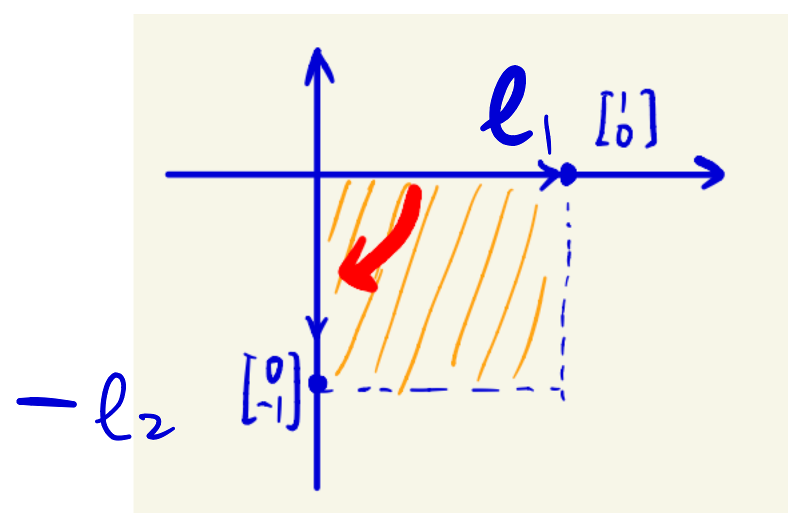
New interpretation: Oriented Area (有向面积)

Geometry Interpretation of Swapping Columns

$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1, \quad \det[e_1, -e_2] = -1.$

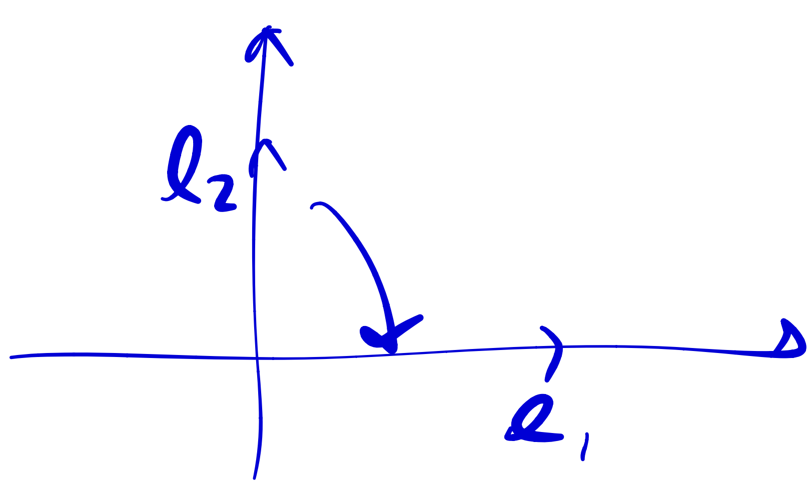
Two column vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ form a square with area 1.

Under the new interpretation: oriented area = -1



$e_1 \longrightarrow -e_2$

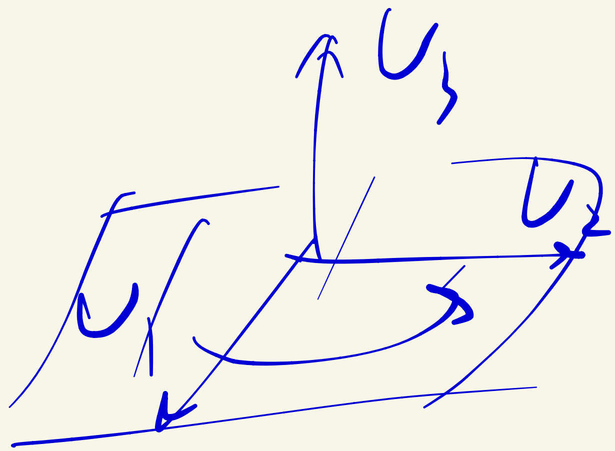
clockwise.



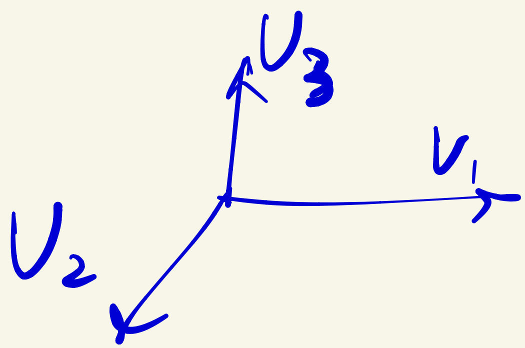
$\det(e_1, -e_2)$

$= \det(e_2, e_1)$

3D Case (orientation)



$\det(e_1, e_2, e_3) = 1.$



$\det(e_2, e_1, e_3) = -1$

Sth. you learned before (NOT in math class).

Right-hand rule,

Biology

Third Term: Swapping Columns Once —> _____

2nd term: $\begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} = \begin{bmatrix} a_{12} & & \\ & a_{21} & a_{23} \\ & a_{31} & a_{33} \end{bmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

3rd term:

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{13} & 0 & 0 \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{vmatrix}$$

$$\vec{a}_1 \quad \vec{a}_2 \qquad\qquad \vec{a}_2 \quad \vec{a}_1$$

$$= -a_{13} \begin{vmatrix} \vec{a}_2 & \vec{a}_1 \end{vmatrix}$$

$$= a_{13} \begin{vmatrix} \vec{a}_1 & \vec{a}_2 \end{vmatrix}$$

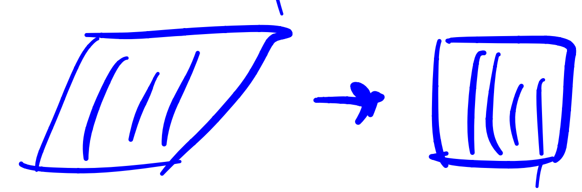
Wrap-up: Compute Determinant for n=3

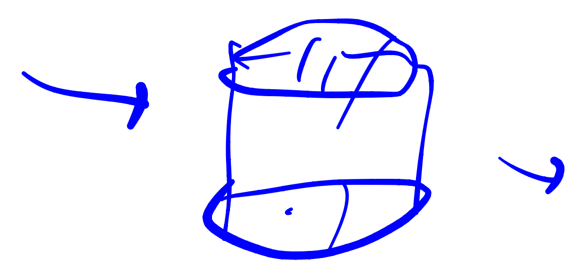
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \overset{①}{=} \begin{vmatrix} a_{11} & 0 & 0 \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

$$\overset{②}{=} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

alternating sign (交错符号)

Key idea.

① Reduce to simpler forms. (orthogonal vec) →  (middle school)

② Reduce to smaller size. →  → height × area (high school)

Key Idea: Reduction

i) Reduce to Simpler Forms;

ii) Reduce to Smaller Size.

Understanding via Simplification Framework

(Req1) Relate A to simpler matrices B_i .

简化

(Req2) Building block: Compute the "volume" of simpler matrices

基本单元

(Req3) **Original-Unit Property**: Relation of det(A) and det(B_i).

原始-单元关系

To generalize: Notation of Cofactor

$$\begin{matrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \qquad \begin{matrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \qquad \begin{matrix} a_{11} & a_{12} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

M_{11} M_{12} M_{13}

Rewrite the formula:

Denote $M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ = matrix obtained by deleting 1st row & 1st column — cofactor of a_{11}

$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ = matrix obtained by deleting 1st row & 2nd column of M — cofactor of a_{12}

$M_{12} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ = — — — — — 1st row & 3rd column of M. — cofactor of a_{13}

$$\det(A) = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|.$$

How to DEFINE det(··)?

Recurse def.

(递归式).

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \times & & & \times \\ \vdots & & & \\ \times & & & \times \end{bmatrix}\right)$$ (volume)

should be $\det(M_{11})\cdot a_{11} - a_{12}\det(M_{12})$ → enough

$+ a_{13}\det(M_{13}) + a_{14}\det(M_{15})$

$\cdots + (-1)^{1+n} a_{1n}\det(M_{1n})$, ①

$$\det(M_{11}) = a_{22}\det(\underbrace{\hat{M}_{22}}) - a_{23}\det(\hat{M}_{23}) + \cdots -$$

same principle as ①) cofactor of a_{22} in $(n-1)\times(n-1)$ matrix M

Recursive in Python / C, ...

$n!$

$$n! = (n-1)! \cdot n.$$

Definition of $\det(A)$

Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix

Denote by $M_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ a matrix formed by deleting the i-th row and j-th column of A, called cofactor of a_{ij}

Definition 18.1 (Determinant)

For a scalar $\alpha \in \mathbb{R}$, define det(α) = α .

For any $A \in \mathbb{R}^{n \times n}$ with $n \geq 2$ define

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} \det(M_{1j}) a_{1j} = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) - - -$$

This is a recursive definition! [递归方式的定义]

Can expand along any row. $= (-1)^{k+1} a_{k1} \det(M_{k1}) + (-1)^{k+2} \det(M_{k2}) a_{k2}$

$+ \cdots$

\underbrace{ M_{ i j}}_{\text{cofactor of } a_{ ij}}

Property: (Expand along any row & col)

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} \det(M_{kj})\, a_{kj}.$$

[This can be proved from the definition; skip the proof].

Properties

It is not hard to verify:

det() defined in Definition 18.1 satisfy the two desired properties.

Property P0 [transpose] $\det(A^{\top}) = \det(A)$.

Property P1 [row/column-linear]
The determinant is a linear function of each row & each col.

Property P2: **[row/column exchange]**
Swapping columns or rows change sign of the determinant.

Examples of the Laplace Expansion

Example

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh$$

6 terms

Examples of the Laplace Expansion

Example

Evaluate

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$$

$$(-1)^5 2 \begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -2 \cdot (-1)^6 3 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \cdot 3 \cdot (10 - 12) = 12$$

$(-1)^{1+4}$

expand along any rows & columns.

Examples of the Laplace Expansion

Example

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{vmatrix} \begin{pmatrix} R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \\ R_5 \to -R_1 + R_5 \end{pmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} (\textit{expand along 3rd column})$$

$$= (-1)^{1+3} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

$$= (1)(1)(2)(3) = 6$$

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Review Questions

How many properties do we have today?
What are they?
Which ones are most nontrivial in your opinion?

Why do we study determinant?
Do you know something that you don’t know before?
What is it?

Advanced questions:
If you were to compute the “volume”, how would you compute it?
Can you think of a different approach (different from using Property P1)?

Summary Today (Write Your Own)

One sentence summary:

We learned determinants.

Detailed summary:

Motivation:

—$ad - bc \neq 0$ is a simple criterion for invertibility.

—To extend it, we notice it indicates "area" of parallelogram.

—Motivating question: how to compute/define "volume" of polytope?

Computing "volume":

—**Overall idea**: "Simplification" framework

—**Properties**: linear over rows/columns; swapping columns changes sign.

—**Definition**: Laplacian expansion over 1st row.

Properties:

—det(AB) = det(A) det(B)

—det(A) $\neq$ 0 iff A is invertible