Lecture 19

Determinant (II) an Linear Transformation (I)

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Today's Lecture: Outline

Main topic: Linear transformation

1. Properties of Determinant

2. Motivation of Linear Transformation

3. Two Definitions of Linear Transformation

Strang's book: Sec 8.1, 8.2

Today's Lecture: Learning Goals

After the lecture, you should be able to

1. Use properties of determinants to compute determinants

2. Describe one application of linear transformation

3. Describe two definitions of linear transformation and explain why they are equivalent

Review

Definition of $\det(A)$

Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix

Denote by $M_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ a matrix formed by deleting the i-th row and j-th column of A

Definition 18.1 (Determinant)

For a scalar $\alpha \in \mathbb{R}$, define det(α) = α .

For any $A \in \mathbb{R}^{n \times n}$ with $n \geq 2$ define

$$\det(A) = \sum_{j=1}^{n} \underbrace{(-1)^{1+j}\det(M_{1j})}_{\text{cofactor of } a_{1j}} a_{1j}$$

This is a recursive definition! [递归方式的定义]

Properties

It is not hard to verify:

det() defined in Definition 18.1 satisfy the two desired properties.

Property P0 [transpose] $\det(A^\top) = det(A)$

Property P1 [row/column-linear]
The determinant is a linear function of each row and each column separately.

Property P2: **[row/column exchange]**
Swapping columns or rows changes the sign of the determinant.

Part I   Properties of det()

Determinant and Invertibility

Proposition 18.1

For any $A \in \mathbb{R}^{n \times n}$, $\det(A) \neq 0$ iff A is invertible.

Geometry:

det(A) = volume of polytope

volume = 0 ⇔ ______________________________

⇔ ______________________________

⇔ ______________________________

Equivalent Conditions for Invertibility ++

Theorem 15.2++ (Equivalent Conditions for Invertibility)

Let $A \in \mathbb{R}^{n \times n}$

The following statements are equivalent:

1. A is **invertible**
2. The linear system $A\mathbf{x} = 0$ has a unique solution $\mathbf{x} = \mathbf{0}$
3. A is a product of elementary matrices
4. A has n pivots; or equivalently: $\mathrm{rank}(A) = n$
5. The columns of A span $\mathbb{R}^n$
6. The columns of A are linearly independent
7. The columns of A form a basis of $\mathbb{R}^n$
8. $\dim(C(A)) = n$
9. $\dim(N(A)) = 0$ or $N(A) = \{\mathbf{0}\}$
10. $\det(\mathrm{A}) \neq 0$

Desired Properties

Property P3 [Diagonal and triangular matrix]

If $A \in \mathbb{R}^{n \times n}$ is a triangular matrix with diagonal entries $a_{11}, \ldots, a_{nn}$

$$\det(A) = a_{11} \ldots a_{nn}$$

Property P4 [product]

For any two matrices $A, B \in \mathbb{R}^{n \times n}$, $\det(AB) = \det(A)\det(B)$

Property P5 [inverse]

For any invertible $A \in \mathbb{R}^{n \times n}$, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Determinants of Type I Elementary Matrices

Type I: **Add a Scaled Row to Another** $(R_j \to \beta R_i + R_j)$

$$E_{\beta R_i + R_j} = \left[\begin{array}{ccc|c|ccc|c|ccc}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
\hline
 & & & 1 & & & & & & & \\
\hline
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
\hline
 & & & \beta & & & & 1 & & & \\
\hline
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{array}\right]
\begin{array}{l}
\\ \\ \\ i\text{th row} \\ \\ \\ \\ j\text{th row} \\ \\ \\ \\
\end{array}$$

ith column jth column

$$\det(E_{\beta R_i + R_j}) = 1 \times 1 \times \cdots \times 1 = 1$$

Determinants of Type I Elementary Matrices

Type I: **Add a Scaled Row to Another** $(R_j \to \beta R_i + R_j)$

$$\det(E_{\beta R_i + R_j}) = 1 \times 1 \times \cdots \times 1 = 1$$

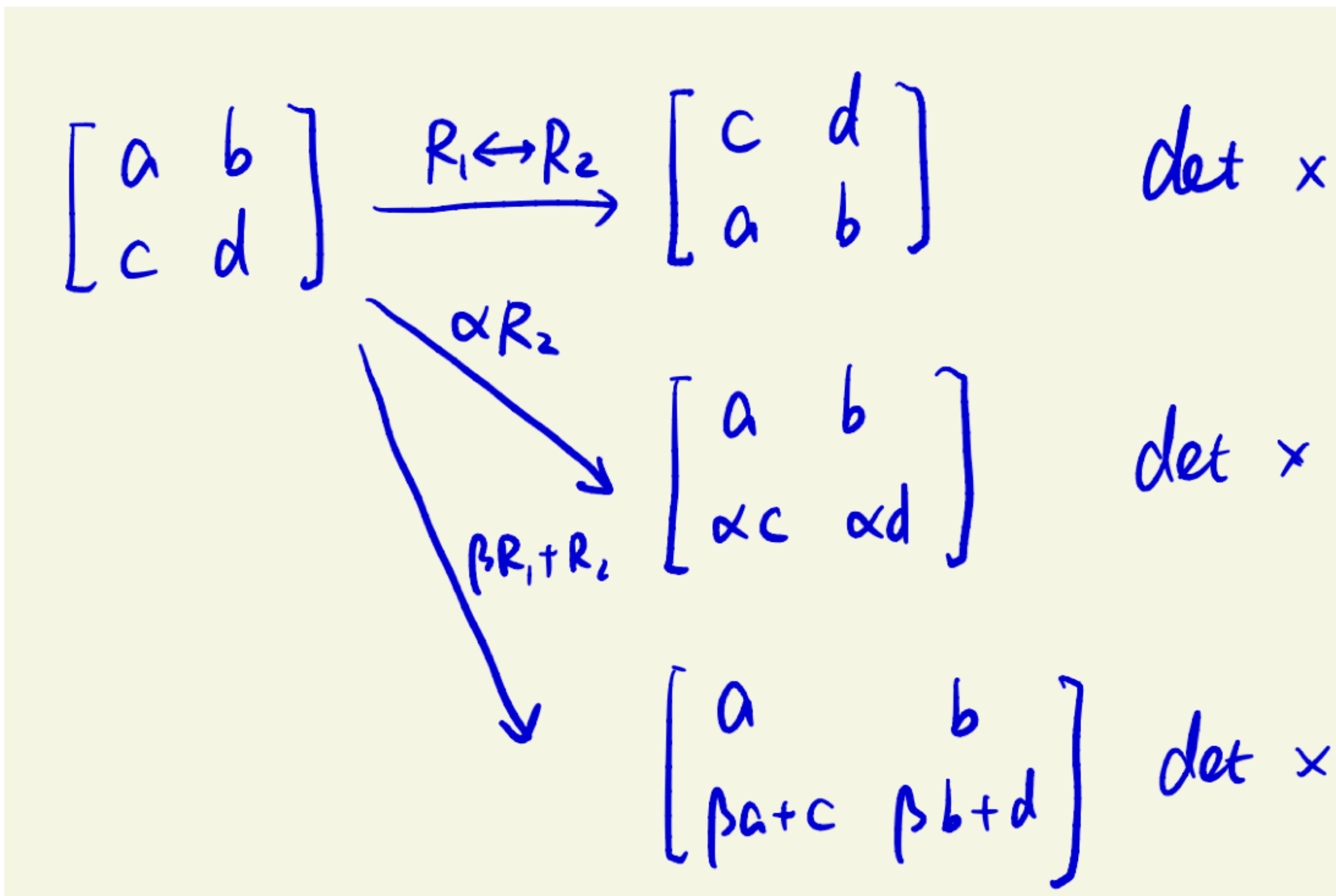
Property P4 [product]

For any two matrices $A, B \in \mathbb{R}^{n \times n}$, $\det(AB) = \det(A)\det(B)$

Property P6 [multiply-add operation]

"Add a Scaled Row to Another" does not change the determinant.

Determinants of Three Elementary Row Operations



Example

Example

$$
\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{vmatrix}
\begin{pmatrix} R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \\ R_5 \to -R_1 + R_5 \end{pmatrix}
$$

$$
= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} (\textit{expand along 3rd column})
$$

$$
= (-1)^{1+3} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix}
$$

$$
= (1)(1)(2)(3) = 6
$$

Reading: Proof Sketch of det(AB) = det(A) det(B)

Two cases.

Case I: A is not invertible. Then AB is not invertible (exercise).
Then det(A)det(B) = 0; and det(AB) = 0. The relation holds.

Case II: A is invertible.
Write $A = E_m \dots E_1$ where each E_i is an elementary matrix.
Lemma: If E is elementary matrix, then det(EK) = det(E)det(K) for any square matrix K.

$$
\begin{aligned}
\det(AB) &= \det(E_m \cdots E_2 E_1 B) \\
&= \det(E_m) \cdot \det(E_{m-1} \cdots E_2 E_1 B) \\
&\vdots \\
&= \det(E_m) \cdot \cdots \cdot \det(E_2) \cdot \det(E_1) \cdot \det(B) \\
&= \det(E_m \cdots E_2 E_1) \cdot \det(B) \\
&= \det(A) \cdot \det(B).
\end{aligned}
$$

Determinants of Type I Elementary Matrices

Property P7 [block matrix]

If A is invertible, then $det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(A)det(D - CA^{-1}B)$.

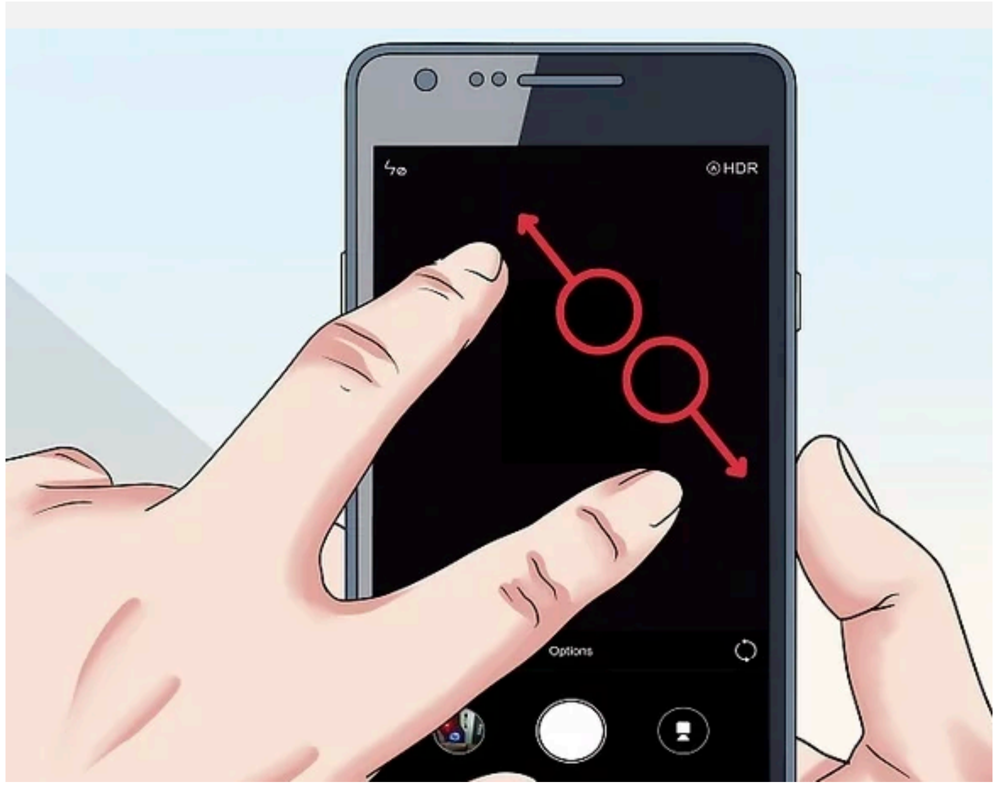
Corollary: det(I + AB) = det(I + BA).

Part II Motivation of Linear Transformation

Zoom Photos

ANDROID » ANDROID APPLICATIONS

How to Zoom with the Camera on Android



Rotate Photos

How to rotate photos on iPhone

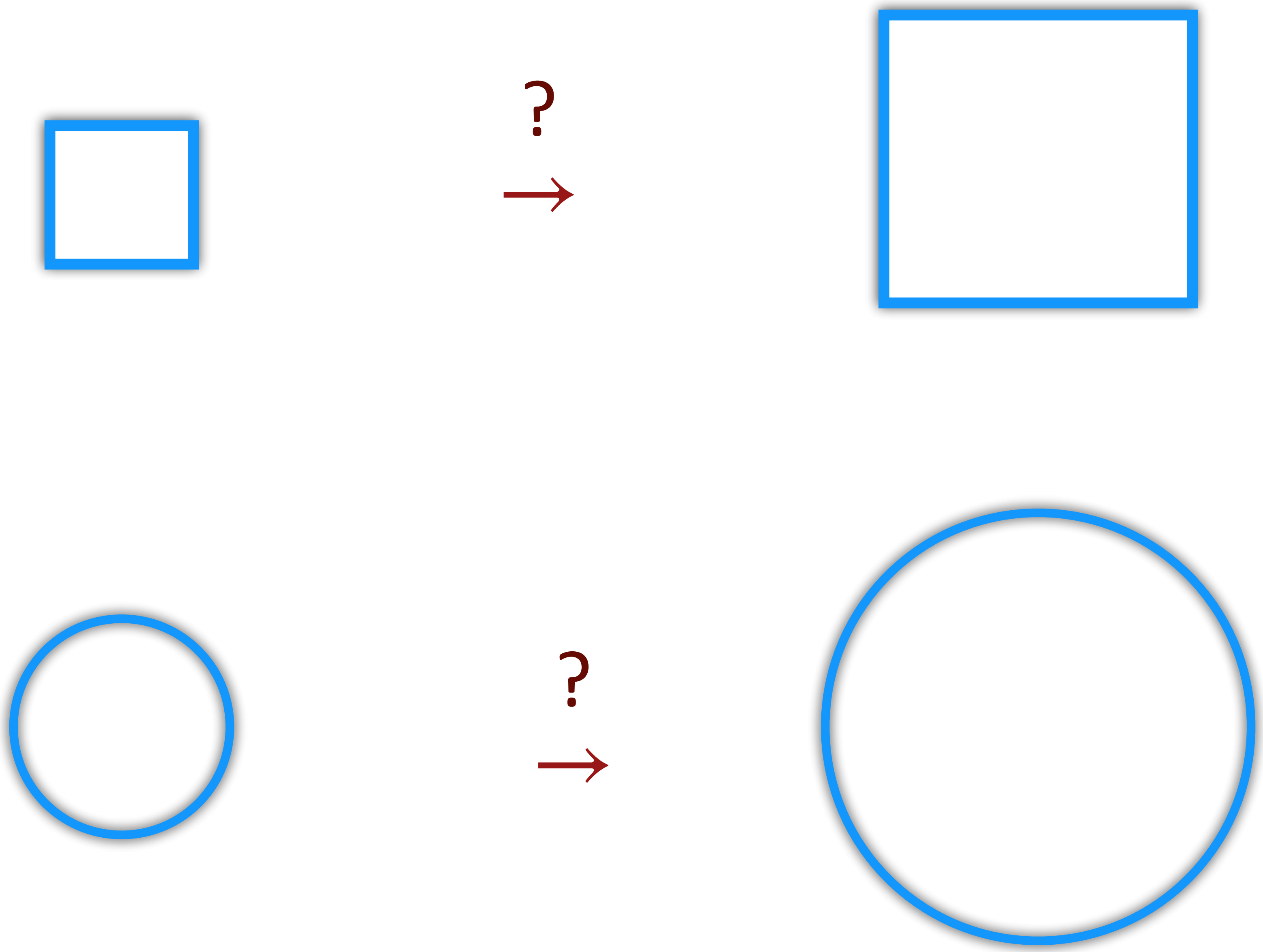
Motivating Question

Yes, I know you can do it on the phones.

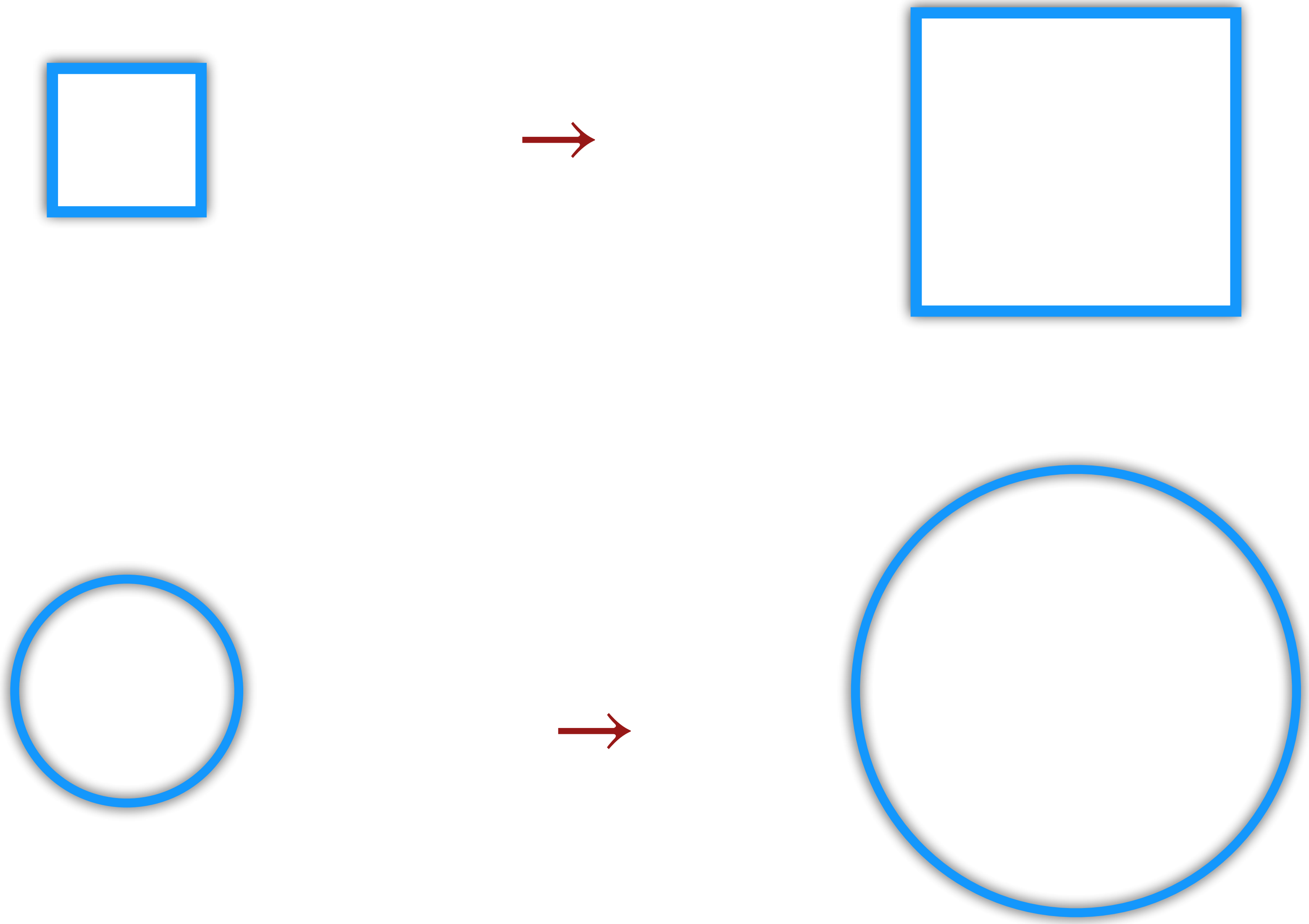
But…

How do the phones accomplish the job?

Zoom Shapes



Zoom Shapes

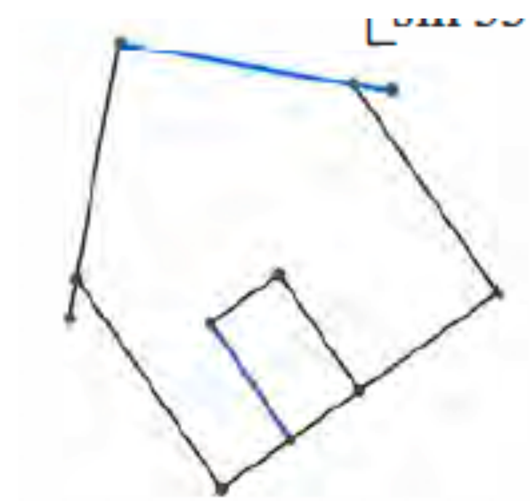


(x,y) → (2x,2y)

Rotating Shapes

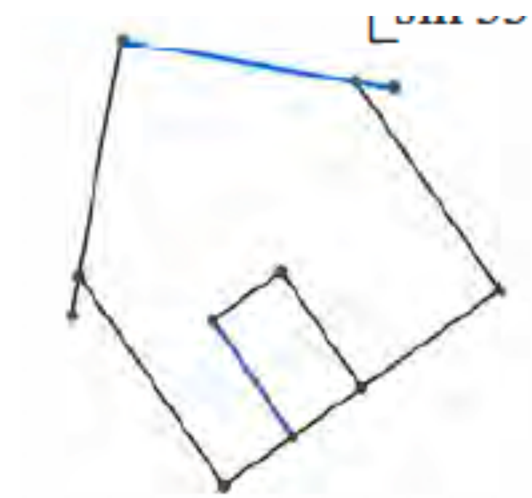
?
→

?
→



Rotating Shapes

 → 

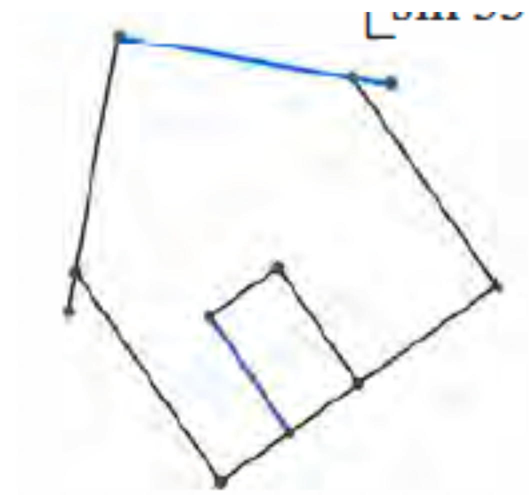
 → 

(x,y) → (x cos 35^o - y sin 35^o,
x sin 35^o + y cos 35^o)

Why true?
How to come up?

Rotating Shapes

→



(x,y) → (x cos 35^o - y sin 35^o, x sin 35^o + y cos 35^o)

Level 1: I memorized: rotation can be achieved by the formula.

Level 2: I can prove: indeed this formula gives the rotation.

Level 3: I know how to derive this formula.

Part III Definition of Linear Transformation

Function —> Mapping

Function (High School):

Chinese: 设两个非空数集X, Y,如果按照某种确定的对应关系f，使得对于X中的任意一个数，在Y中都有唯一确定的一个数f(x)与之对应，那么就称f：X —> Y是一个函数。

English: Given two non-empty sets of numbers, X and Y, if there exists a certain definite correspondence relation f, such that for every number in set X, there is a uniquely determined number f(X) in set Y corresponding to it, then this relation f: X —> Y is called a function.

Mapping:

A mapping from a set X to a set Y assigns to each element of X exactly one element of Y.
The set X is called the domain of the mapping and the set Y is called the codomain of the mapping.

Remark: In math, many people use mapping and function interchangeably. Some people, e.g., Serge Lang, requires the codomain of a function to be a set of numbers.

Linear Function

Definition 19.1:

Suppose $a_1, \ldots, a_n$ are given real numbers, $\mathbf{x} = (x_1, \ldots, x_n)$.

$f(\mathbf{x}) = a_1 x_1 + \ldots + a_n x_n$ is called a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.

Linear Function

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$f(\mathbf{x}) = a_1x_1 + \ldots + a_nx_n$ is called a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.

Equivalent Definition:

Suppose $\mathbf{a} \in \mathbb{R}^n$ is a given real vector, $\mathbf{x} \in \mathbb{R}^n$ . .

$f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle$ is called a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.

Recall: From Lecture 3:

Definition of linear equation:

Suppose $a_1, \ldots, a_n, b$ are given real numbers, $x_1, \ldots, x_n$ are variables.

We say $a_1x_1 + \ldots + a_nx_n = b$ is a linear equation.

Linear Transformation (Euclidean space)

Definition 19.2:

Suppose $a_{i,j}, i = 1,\ldots,m; j = 1,\ldots,n$ are given real numbers.

$$f(\mathbf{x}) = (a_{11}x_1 + \ldots + a_{1n}x_n,\ \ldots,\ a_{m1}x_1 + \ldots + a_{mn}x_n)$$

is called a linear transformation (or linear map) from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Equivalent Definition (matrix form):

Suppose $A \in \mathbb{R}^{m\times n}$ is a given real matrix.

$f(\mathbf{x}) = A\mathbf{x}$ is called a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Linear Function v.s. Linear Transformation

$f(x) = a^T x$ linear function. $\mathbb{R}^n \to \mathbb{R}$

$$f(x) = \begin{pmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{pmatrix}$$ linear transformation. $\mathbb{R}^n \to \mathbb{R}^m$

Recall: From Lecture 3:
Definition of linear system of equations:

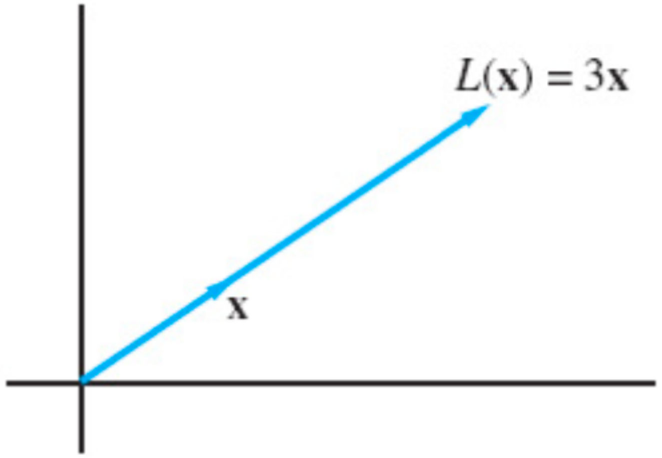
Suppose $a_{i,j}, i = 1,\ldots,m; j = 1,\ldots,n$ are given real numbers.

We say $a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \;\; \ldots, a_{m1}x_1 + \ldots + a_{mn}x_n = b_n$ is a system of linear equations.

Examples of Linear Transformations

Example

$L : \mathbb{R}^2 \to \mathbb{R}^2$. Stretching or shrinking: $L((x, y)^T) = (\alpha x, \alpha y)^T (\alpha > 0)$

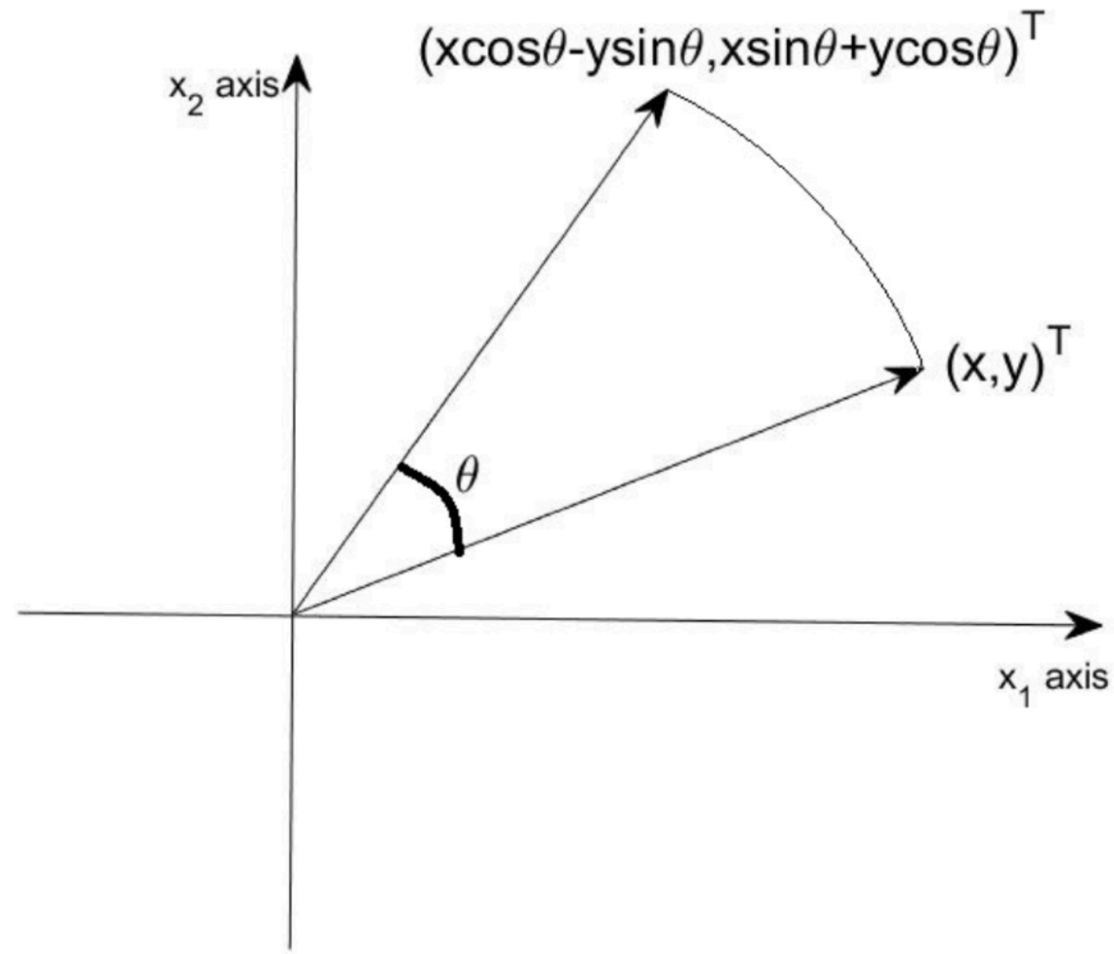


Examples of Linear Transformations

Example

$L : \mathbb{R}^2 \to \mathbb{R}^2$. Rotation:

$L((x, y)^T) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)^T$. (Rotate in anticlockwise by angle θ)



Examples of Linear Transformations

Example

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as:

$$L((x, y)^T) = x - y$$

Relation and Difference?

Linear transformation	Linear system of equations
$f(\mathbf{x}) = A\mathbf{x}$	$A\mathbf{x} = \mathbf{b}$

Math form: similar

Meaning: different

Mapping	Equation
Describes a relation	Built on mapping

e.g. President is a "relation"

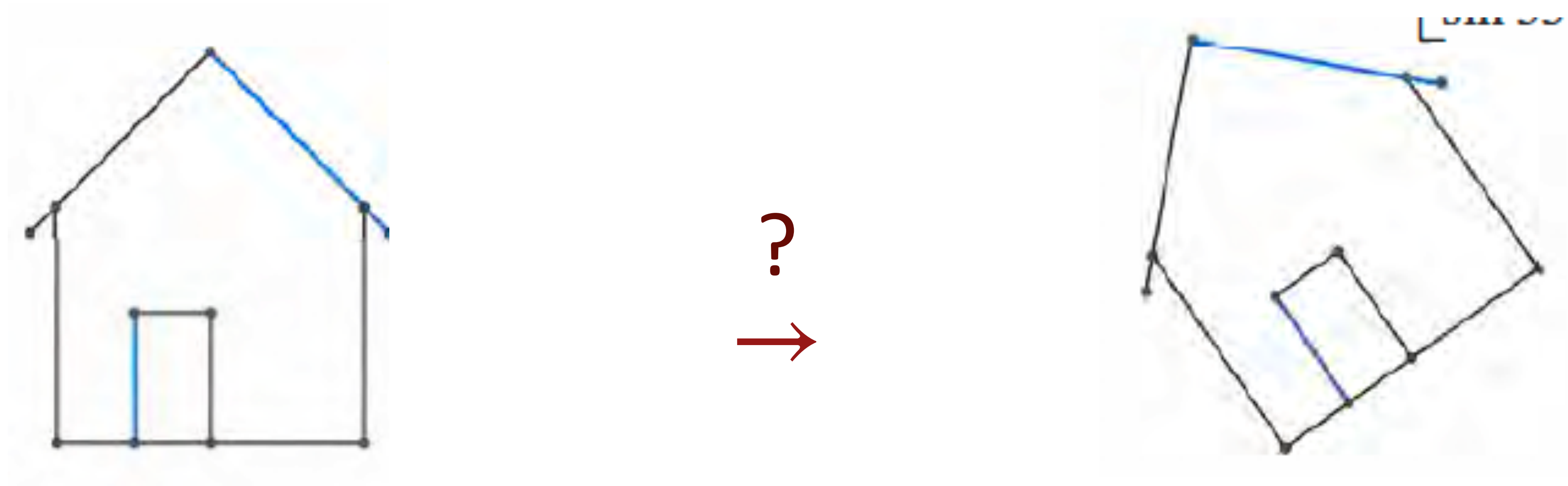
equation is "question"

f(?) = Putin?

Relation and Difference?

	Linear transformation	Linear system of equations
Math form	$\mathbf{y} = A\mathbf{x}$	$A\mathbf{x} = \mathbf{b}$
Applications	Rotating images	Compute price; Compute references…
Known	$\mathbf{x}$; “description" of $\mathbf{y}$	A, $\mathbf{b}$
Ultimate goal	Compute $\mathbf{y}$	Compute $\mathbf{x}$
Intermediate	obtain A	

Rotating Shapes



Rotating is a “transformation”.

Question 1: How do I know it’s a linear transformation?

Question 2: How do I derive its expression?

Remark: I might ask you, what questions d

Part IV Another Definition of Linear Transformation

Linear Transformation ==> Superposition (叠加)

Property 19.1 [superposition property]

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

In words, transformed LC of vectors = LC of transformed vectors.

Examples:

Non-examples:

Linear Transformation ==> Superposition (疊加)

Property 19.1 [superposition property]

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n .$$

In words, transformed LC of vectors = LC of transformed vectors.

Corollary 19.1

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}) .$$

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n .$$

Superposition ==> Linear Transformation

Definition:

Suppose $f_i(\mathbf{x})$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$, $i = 1,\ldots,m$.

$f(\mathbf{x}) = (f_1(\mathbf{x}),\ \ldots,\ f_m(\mathbf{x}))$ is called a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Remark: Can also call it "vector function" (向量函数).

Theorem 19.1

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proof for m = 1

Proposition 19.1 (n=1 case for Thm 19.1)

If a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

then we must have $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x}$ for some $\mathbf{a}$.

Exercise 1: Prove that if $f: \mathbb{R} \to \mathbb{R}$ satisfies $(*)$ for $n=1$, then $f(x) = ax$, $\forall x \in \mathbb{R}$ for some $a \in \mathbb{R}$.

Hint: What is $f(2)$? $f(3)$? $f(2.5)$?

Exercise 2: Prove that if $f: \mathbb{R}^2 \to \mathbb{R}$ satisfies $(*)$, then $f(x) = a_1 x_1 + a_2 x_2$ for some $a_1, a_2 \in \mathbb{R}$.

Proof for m=1, n=1

Analysis:

Conditions

$$\begin{cases} f(\alpha x) = \alpha f(x) \\ f(x+y) = f(x) + f(y) \end{cases}$$

Conclusion

Want $f(x) = ax$ for some a.

Hint: What is $f(2)$? $f(3)$? $f(2.5)$?

What do we hope $f(2)$, $f(3)$, $f(2.5)$ to be?

Proof for m=1, n=2

Conditions: $f(\alpha x) = \alpha f(x)$. ① $f(x+y) = f(x) + f(y)$. ②

Want: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = a_1 x_1 + a_2 x_2$ for some a_1, a_2.

Analysis. $f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$, $f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = ?$ ~~$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2 f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$~~

$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \overset{②}{=}$

$f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) =$

Proof: Denote

Then $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

Proof for m=1, general n

Conditions: $f(\alpha x) = \alpha f(x)$. ① $f(x+y) = f(x) + f(y)$. ② $f: \mathbb{R}^n \to \mathbb{R}$.

Want: $f(x) = a^T x$, $\forall x \in \mathbb{R}^n$, for some $a \in \mathbb{R}^n$.

Proof: $f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) =$

Proof for general m, general n

Conditions: $f(\alpha x) = \alpha f(x)$. ① $f(x+y) = f(x) + f(y)$. ②. $f: \mathbb{R}^n \to \mathbb{R}^m$.

Want: $f(x) = Ax$, $\forall x \in \mathbb{R}^n$, for some $A \in \mathbb{R}^{m \times n}$.

Proof: Suppose $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$.

From ①, ②, we get

$f_i(\alpha x) =$ $\quad\quad$ $f_i(x+y) =$

From Prop. 19.1, we get:

Denote $A =$

Then $f(x) =$

Another Definition

Definition 19.2 (alternative definition of LT):
If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies
$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$
Then we say f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Equivalent definition to Def 19.1.

Using properties to define sth.

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Summary Today (Instructor)

One sentence summary:

We learned definition and derivation of linear transformation

Detailed summary:

Properties of Determinants:

—det(AB) = det(A) det(B)

—det(A) $\neq$ 0 iff A is invertible

Motivation/application of Linear transformation:

—How to zoom/rotate photos?

Definitions of Linear transformation (Euclidean space)

Def 1: f(x) = Ax

Def 2: Any f that satisfies $f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.