Lecture 20

Linear Transformation II

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Today's Lecture: Outline

Main topic: Linear transformation II

1. Two Definitions of Linear Transformation

2. How to Derive Expression of Linear Transformation

3. Linear Transformation on General Linear Spaces

Strang's book: Sec 8.1, 8.2

Today's Lecture: Learning Goals

After the lecture, you should be able to

1. Tell two definitions and why they are equivalent
2. Verify linear transformation and compute it for Euclidean spaces
3. Tell the relation of linear transformation and matrix
4. Verify linear transformation and compute it for general linear spaces

Midterm Exam ~~Design Idea~~ Comments

语文 Chinese & English

Midterm Exam Common Issues

$\underbrace{C(A)}_{\text{space}} = \underbrace{c\_1\ a\_1 + \ldots + c\_n\ a\_n}_{\text{vector}}$

Issue: Confuse space and vectors.

CUHKSZ = professor.

= Prof. Xu.

$\underbrace{C(A)}_{\text{space}} = \underbrace{n - \dim(N(A))}_{\text{number}}$.

Issue: LHS is a space， RHS is "dimension" which is a number.

Cannot be the same

→ prize = 3 yuan

→ ~~prize = 3~~

~~CUHKSZ = 11,000~~

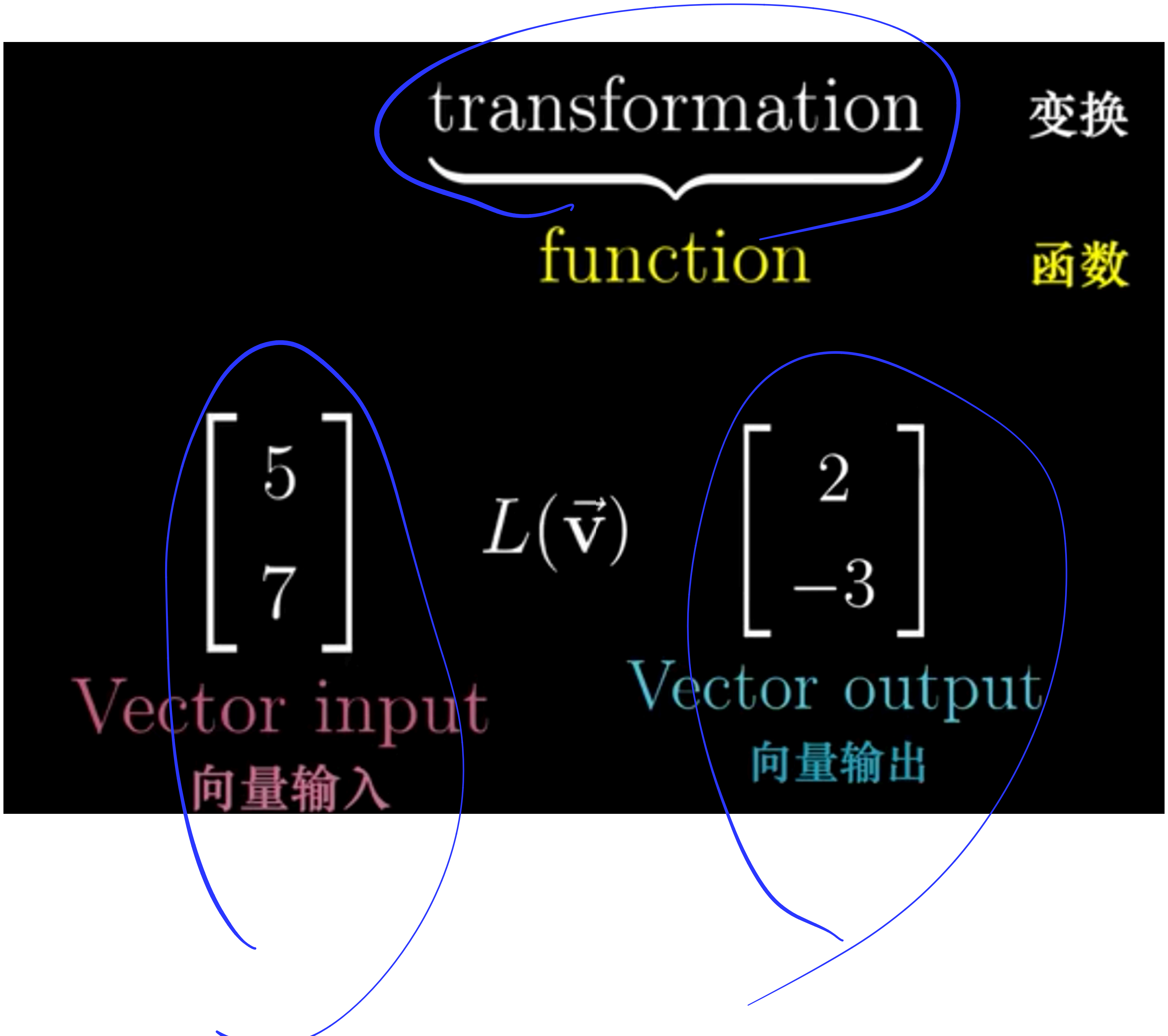
CUHKSZ has 11,000 ppl.

3 $\underbrace{C([A,B])}_{\text{space}} = \underbrace{\{ a\_1 , \ldots, a\_n, b\_1, \ldots, b\_n\}}_{\text{set}}$

space = $\text{span}(\{a_1, \ldots, a_n, b_1, \ldots, b_n\})$ ∞-size set

Review

Transformation: Vector Input, Vector Output



Rotate Photos

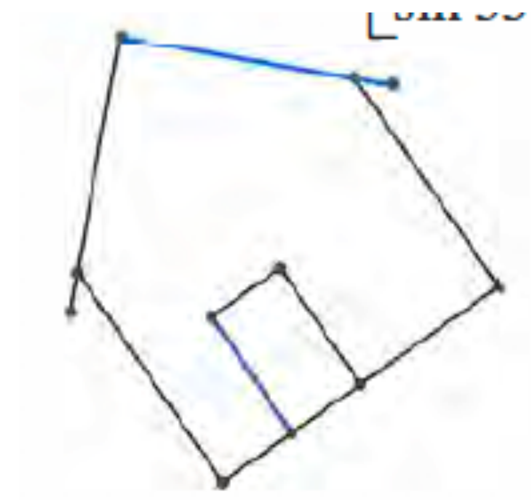
How to rotate photos on iPhone

Motivating Question

Yes, I know you can do it on the phones.

How do the phones accomplish the job?

? →



Question 1: Is it a linear transformation?

Question 2: How do I derive its expression?

Part I Another Definition of Linear Transformation

Linear Transformation ==> Superposition (叠加)

Property 19.1 [superposition property]

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha,\beta \in \mathbb{R}, \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n. \quad (*)$$

f of LC = LC of f

In words, transformed LC of vectors = LC of transformed vectors.

Examples: $f(x) = 2x$,

$$f(\alpha x+\beta y) = 2(\alpha x+\beta y) = \alpha\cdot(2x) + \beta\cdot(2y) = \alpha f(x)+\beta f(y)$$

Non-examples: $f(x) = x^2$,

$$f(\alpha x+\beta y) = (\alpha x+\beta y)^2 = \alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta xy$$

$$\neq \alpha x^2 + \beta y^2 \quad (= \alpha f(x)+\beta f(y))$$

Linear Transformation ==> Superposition (叠加)

Property 19.1 [superposition property]

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\alpha\mathbf{x}+\beta\mathbf{y})=\alpha f(\mathbf{x})+\beta f(\mathbf{y}), \quad \forall\alpha,\beta\in\mathbb{R}, \forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n. \quad (*)$$

In words, transformed LC of vectors = LC of transformed vectors.

Corollary 19.1

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\alpha\mathbf{x})=\alpha f(\mathbf{x}). \quad ①$$

$$f(\mathbf{x}+\mathbf{y})=f(\mathbf{x})+f(\mathbf{y}), \forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n. \quad ②$$

$(*) \Longleftrightarrow ① \,\&\, ②$

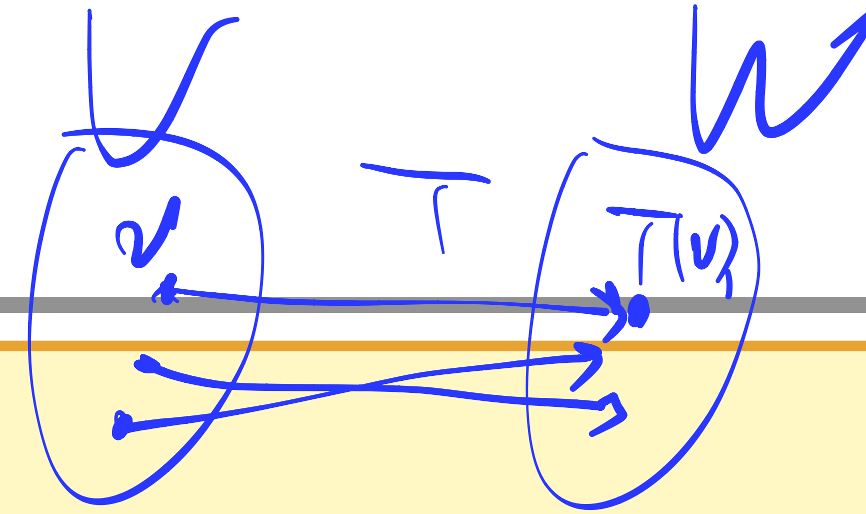
$f(x)=2x$

$f(\alpha x)=2\alpha x=\alpha f(x)$

$f(x+y)=2(x+y)$

$=2x+2y=f(x)+f(y)$

Mapping



Definition 20.1 (Mapping)

Let V and W be two sets.

If for any $v \in V$, there is a unique $w = T(v) \in W$, then we say T is a **mapping** from V to W.

V is called the **domain** of the mapping.

W is called the **codomain** of the mapping.

Special case:

Suppose $f_i(\mathbf{x})$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$, $i = 1, \ldots, m$.

$f(\mathbf{x}) = (f_1(\mathbf{x}), \ \ldots, f_m(\mathbf{x}))$ is called a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Remark: Can also call it "vector function" (向量函数).

$LT \Rightarrow S.P.$

Superposition ==> Linear Transformation

Theorem 20.1

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$S.P \Rightarrow LT.$

$f(x) = A \cdot x$

A: $m \times n$, x: $\mathbb{R}^n$

Proof for m = 1

Proposition 20.1 (n=1 case for Thm 20.1)

If a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

then we must have $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x}$ for some $\mathbf{a}$.

1×1

Exercise 1: Prove that if $f: \mathbb{R} \to \mathbb{R}$ satisfies $(*)$ for $n=1$, then $f(x) = ax$, $\forall x \in \mathbb{R}$ for some $a \in \mathbb{R}$.

Hint: What is $f(2)$? $f(3)$? $f(2.5)$?

Exercise 2: Prove that if $f: \mathbb{R}^2 \to \mathbb{R}$ satisfies $(*)$, then

2×1

$f(x) = a_1 x_1 + a_2 x_2$ for some $a_1, a_2 \in \mathbb{R}$.

Proof for m=1, n=1

Known | Goal

Analysis,

Conditions

$$\begin{cases} f(\alpha x) = \alpha f(x) & ① \\ f(x+y) = f(x) + f(y) & ② \end{cases}$$

Conclusion

Want $f(x) = ax$ for some a.

Hint: What is $f(2)$? $f(3)$? $f(2.5)$? | What do we hope $f(2), f(3), f(2.5)$ to be ?

$f(2) = 2 \cdot f(1)$.

$f(3) = 3 \cdot f(1)$

$f(2.5) = 2.5 f(1)$

Bridge

$f(2) = 2a$

$f(3) = 3a$

$f(2.5) = 2.5a$

for some a

Let $a \triangleq f(1)$, then

Proof: $f(x) = f(x \cdot 1) \overset{①}{=} x f(1) = ax$,

where $a \triangleq f(1)$. $\square$

Proof for m=1, n=2

Conditions: $f(\alpha x) = \alpha f(x)$. ① $f(x+y) = f(x) + f(y)$. ②

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ (*)

Want: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = a_1 x_1 + a_2 x_2$ for some a_1, a_2.

Analysis. $f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = ?$ ~~$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2 f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$~~

$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \overset{②}{=} f\left(1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1 f(e_1) + 2 f(e_2)$

$f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = f(2e_1 + 3e_2) = 2f(e_1) + 3f(e_2)$

$\boxed{\text{Key}}$

basis representation

$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Proof: Denote

Then $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f(x_1 e_1 + x_2 e_2)$

$\overset{(*)}{=} x_1 f(e_1) + x_2 f(e_2)$

$= a_1 x_1 + a_2 x_2$, where $a_1 \triangleq f(e_1), a_2 \triangleq f(e_2)$

Proof for m=1, general n

Conditions: $f(\alpha x) = \alpha f(x)$. ① $f(x+y) = f(x) + f(y)$. ②. $f: \mathbb{R}^n \to \mathbb{R}$.

Want: $f(x) = a^T x$, $\forall x \in \mathbb{R}^n$, for some $a \in \mathbb{R}^n$.

Proof: $f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = f(x_1 e_1 + \cdots + x_n e_n)$

$$= x_1 f(e_1) + \cdots + x_n f(e_n)$$

$$= a_1 x_1 + \cdots + a_n x_n,$$

where $a_i = f(e_i)$, $\forall i$.

Proof for general m, general n

Conditions: $f(\alpha x) = \alpha f(x)$. ① $f(x+y) = f(x) + f(y)$. ②. $f: \mathbb{R}^n \to \mathbb{R}^m$

Want: $f(x) = Ax$, $\forall x \in \mathbb{R}^n$, for some $A \in \mathbb{R}^{m \times n}$.

Proof: Suppose $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$.

From ①, ②, we get (for each $f_i(x)$)

$$f_i(\alpha x) = \alpha f_i(x), \quad f_i(x+y) = f_i(x) + f_i(y)$$

From Prop. 20.1, we get:

$$f_i(x) = a_i^T x, \text{ for some } a_i \in \mathbb{R}^{n \times 1}.$$

Denote $A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$

Then $f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} = A \cdot x$. $\square$

Another Definition

Definition 20.2 (alternative definition of LT):

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then we say f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Equivalent definition to Def 20.1.

Using properties to define sth.

Linear Transformation in Euclidean Space: Two Definitions

Two equivalent definitions:

ax

$\downarrow$

Ax

Definition 19.1 (matrix form):

Suppose $A \in \mathbb{R}^{m \times n}$ is a given real matrix.

$f(\mathbf{x}) = A\mathbf{x}$ is called a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 20.1 (alternative definition of LT):

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then we say f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Why Equivalent

$Ax \Rightarrow$ Property def

Property 20.1 [Def 19.1 ==> Def 20.1]

Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is defined as $f(\mathbf{x}) = A\mathbf{x}$, then

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n .$$

Reverse: Property Def $\longrightarrow$ Ax def.

Theorem 20.1 [Def 20.1 ==> Def 19.1]

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then $f(\mathbf{x}) = A\mathbf{x}$ for some matrix A .

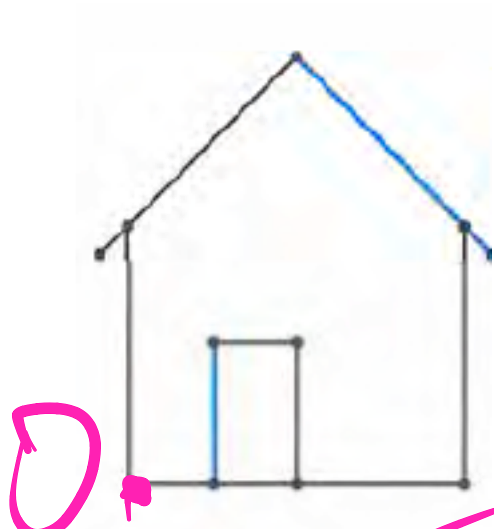
rank. $\to$ def. # of pivots

$\to$ def. dim (Col span).

Part II Expression of Linear Transformation

Transformation: Movement

Description of "Rotation"



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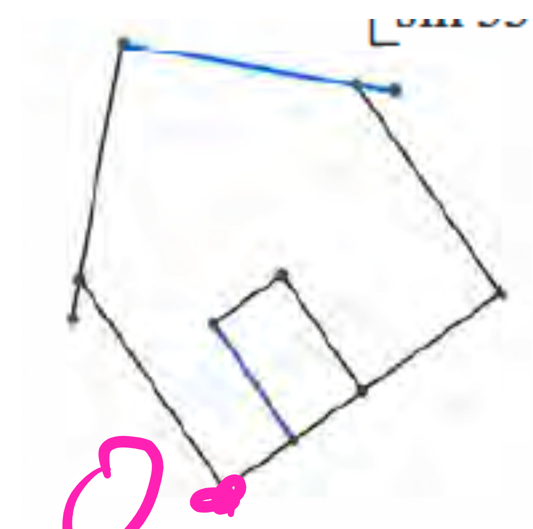


Rotation is a "transformation".

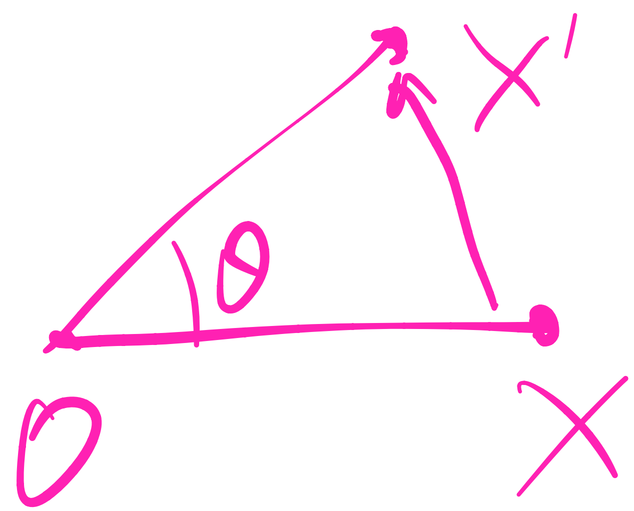
expression

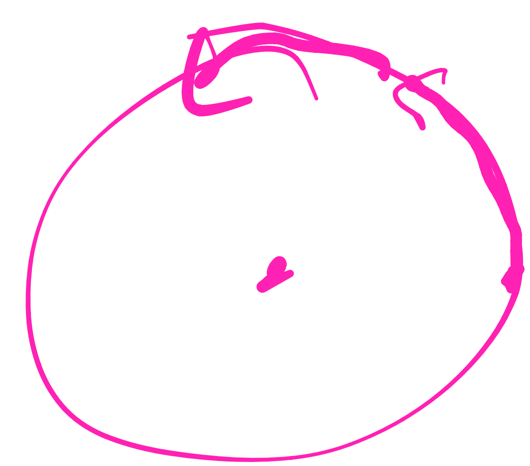
Precise description of rotation:

"Rotating an image around point O by degree θ" means:

For any point X, the new point X' satisfies $\angle(OX, OX') = \theta$.

$|OX| = |OX'|$



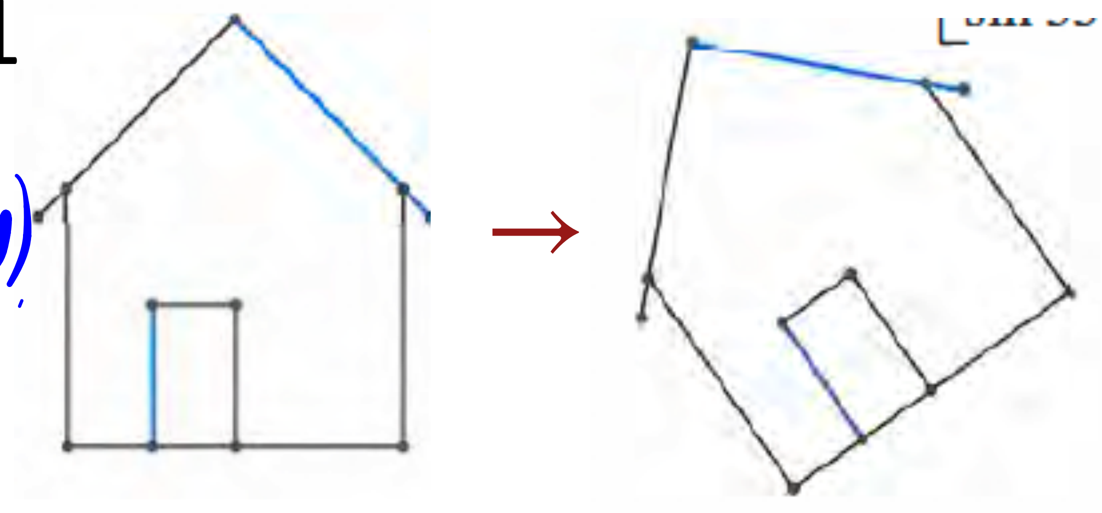


Q1 How to Check Linear Transforrmation

Question 1: Is it a linear transformation?

Check: It satisfies Property 19.1

$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$



algebraic

geometrical way of checking

Checking Linear Transformation: Intuition

Question 1: Is it a linear transformation?

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

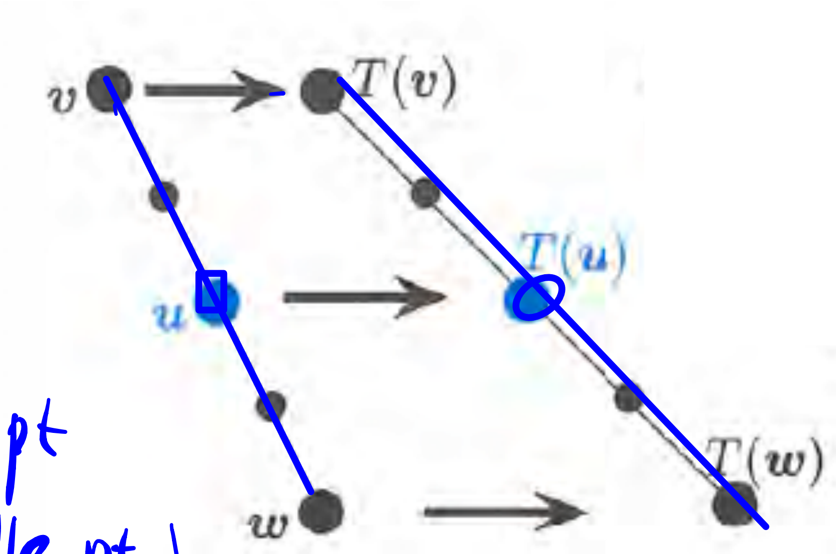
Fix $\mathbf{x}, \mathbf{y}$, consider all $\alpha, \beta \in \mathbb{R}$.

$\alpha + \beta = 1$

Rule 1: Lines to Lines.

Rule 2: Equally spaced points to equally space points.

(middle pt → middle pt)



$\alpha = \beta = 0$

Rule 3: Origin to Origin.

$f(0) = 0 + 0 = 0.$

Derivation of "Lines to Lines"

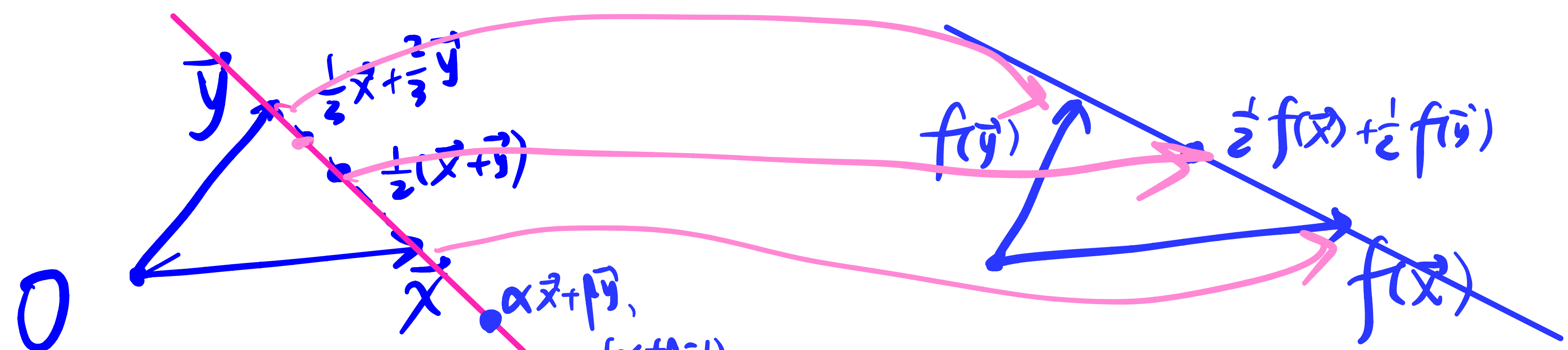
The superposition property implies

$$\{f(\alpha\vec{x}+\beta\vec{y}) : \alpha+\beta=1\} = \{\alpha f(\vec{x})+\beta f(\vec{y}) ; \alpha+\beta=1\} \quad (1)$$

$$(1) \iff \Omega_1 \xrightarrow{f} \Omega_2 \overset{(2)}{\iff} \text{line} \xrightarrow{f} \text{line}.$$

$\Omega_1 \triangleq \{\alpha\mathbf{x}+\beta\mathbf{y} \mid \alpha+\beta=1\}$ $\qquad$ $\Omega_2 \triangleq \{\alpha f(\mathbf{x})+\beta f(\mathbf{y}) \mid \alpha+\beta=1\}$

To show: Ω_1 = line (2) $\qquad$ Ω_2 = line. (2)



Geometrical interpretation of (1) is

$$\text{line} \xrightarrow{f} \text{line}$$

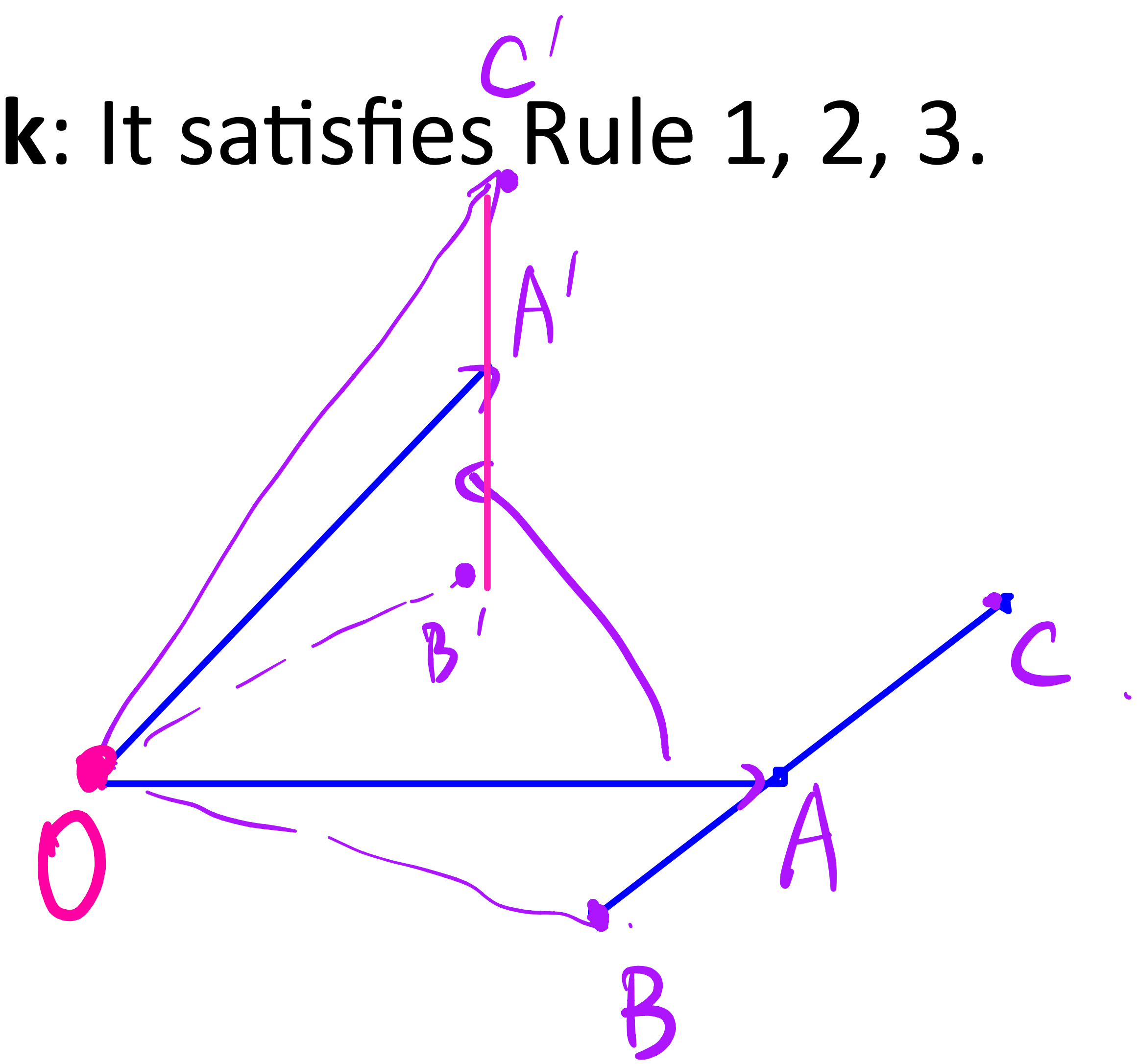
$$\text{middle pt} \xrightarrow{f} \text{middle point}$$

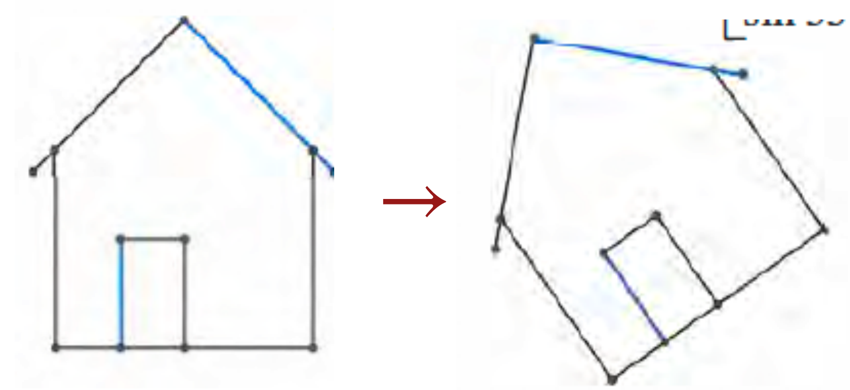
Use 3 Rules to Check

Rule: $O \to O$.

Question 1: Is it a linear transformation?

Check: It satisfies Rule 1, 2, 3.





$A \xrightarrow{\text{rotate } \theta} A'$

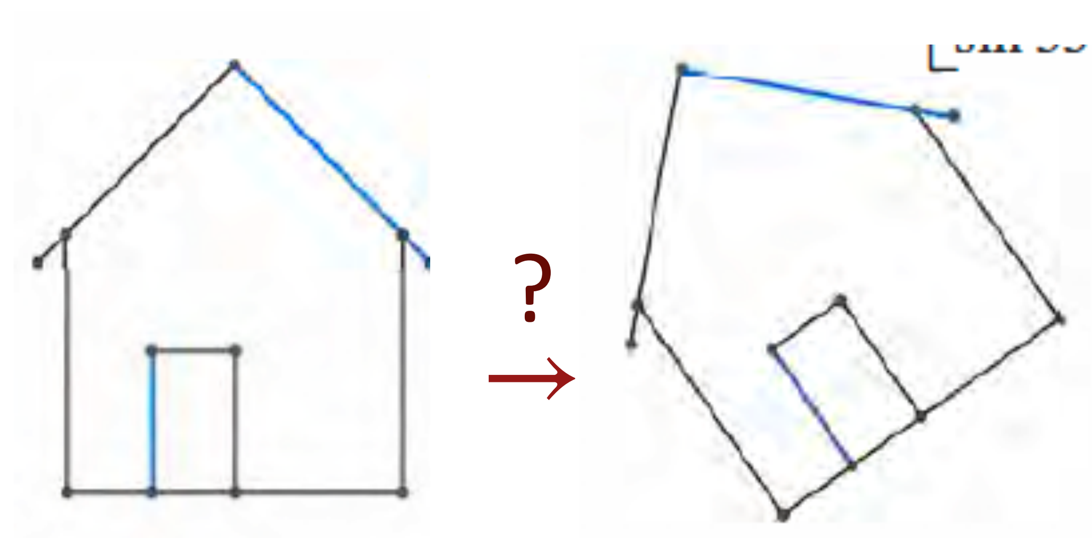
$B \longrightarrow B'$

$C \longrightarrow C'$

① A', B', C', same line. (can prove by similar triangles, 全等三角形)

② A' is the middle point of B'C'.

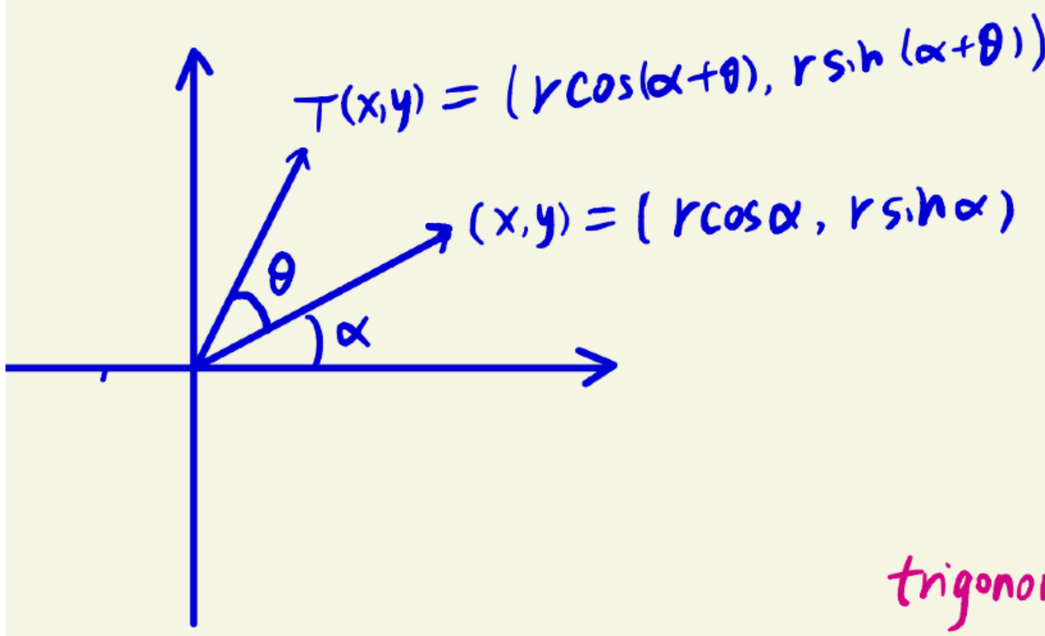
Q2: Math Form of Rotation



Q2: How to derive its expression?

Reading: Direct Derivation (high school student)

Method 1.



What's the relation of $T(x,y)$ and (x,y)?

$(x,y) = (r\cos\alpha, r\sin\alpha)$ for some α, some $r>0$.

$$T(x,y) = \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$$

trigonometric identities [三角恒等式]

$$= \begin{bmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{bmatrix}$$

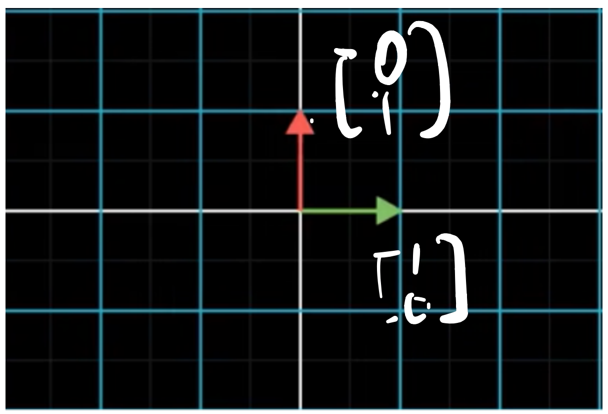
Need:
① (x,y)
$\downarrow$
$T(x,y)$
② (linear)

New method (LT)
① $T(e_1), T(e_2)$
② $A = [T(e_1), T(e_2)]$

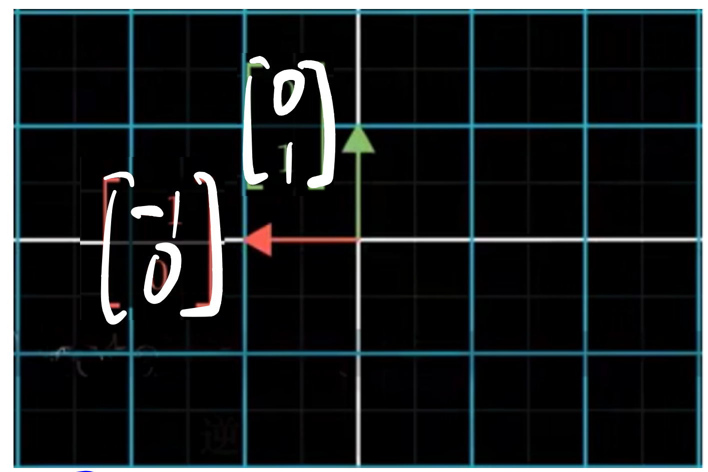
<u>Remark</u> This method requires knowing the trigonometric identities.
We'll provide a more general method next.

Special Case: Rotating by 90 degree

Example: Rotation by 90^o $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



$\longrightarrow$



$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{\text{rot } 90^\circ} T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{\text{rot } 90^\circ} T(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Proof 20 mins ago

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow T(x) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

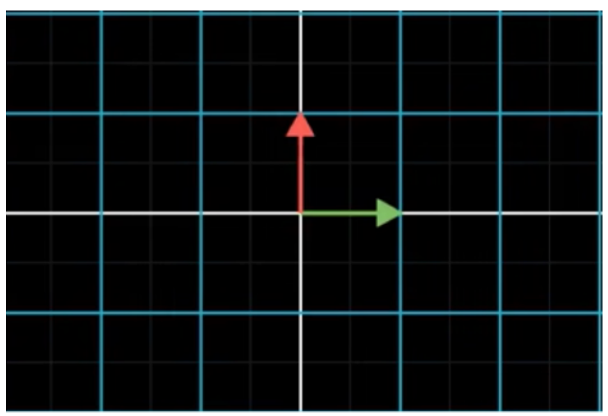
$$= x_1 T(e_1) + x_2 T(e_2)$$

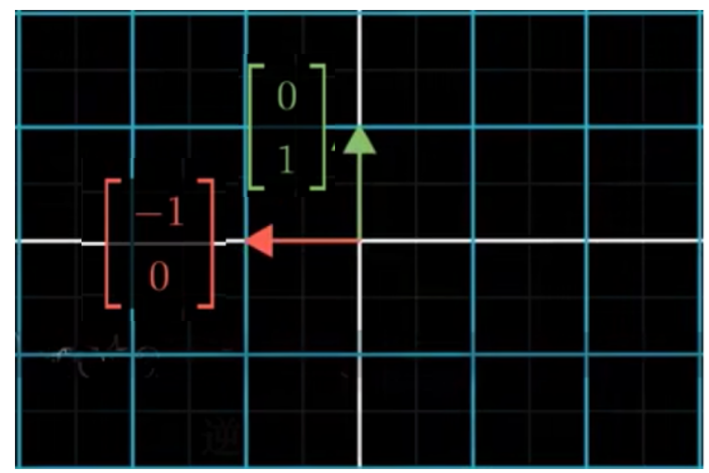
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot x_1 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} x_2$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [T(e_1), T(e_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Special Case: Rotating by 90 degree

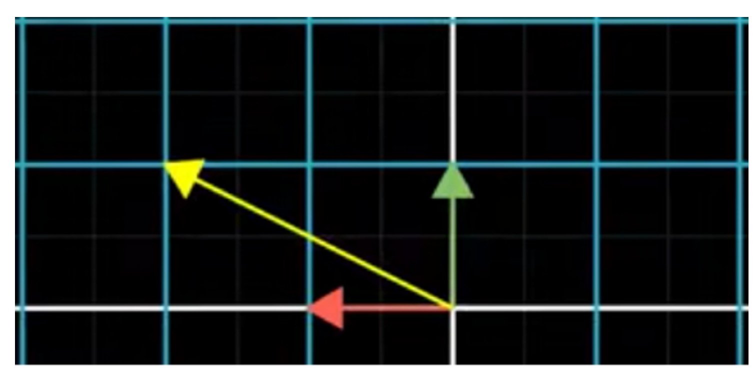
Example: Rotation by 90^o $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$





basis (e_1, e_2) $\longrightarrow$ basis $(T(e_1), T(e_2))$

(1,2)



$x = (e_1, e_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $T(x) = (T(e_1), T(e_2))\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

basis change, coordinate same.

LT: $x \to y = Ax$.

1) change basis

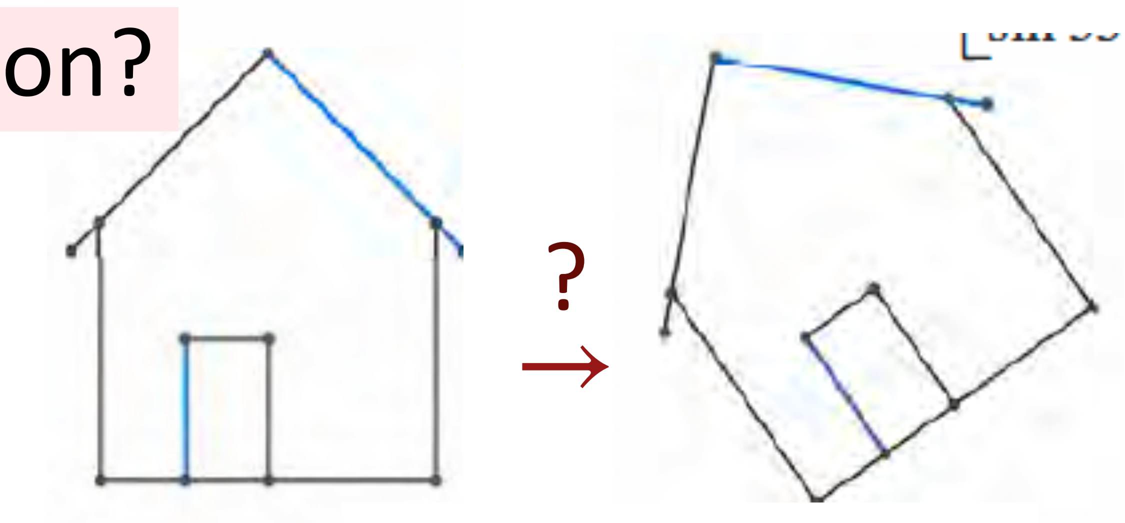
$e_1, \ldots, e_n \longrightarrow T(e_1), \cdots, T(e_n)$.

2) coordinate stay

$y = [T(e_1), \ldots, T(e_n)]\vec{x}$.

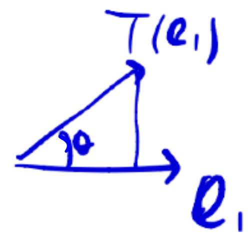
Q2: Math Form of Rotation

Q2: How to derive its expression?

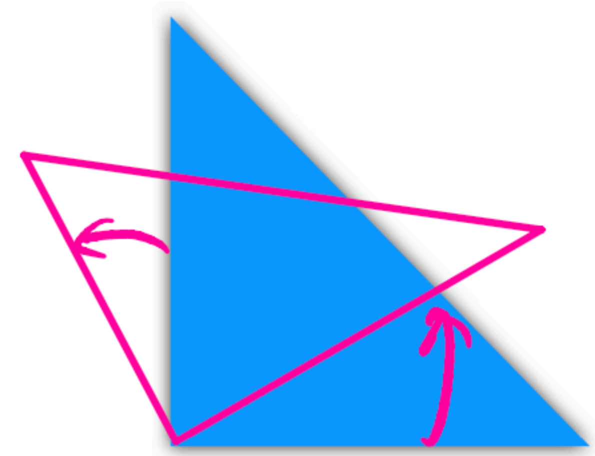


From basis!

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto T(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto T(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

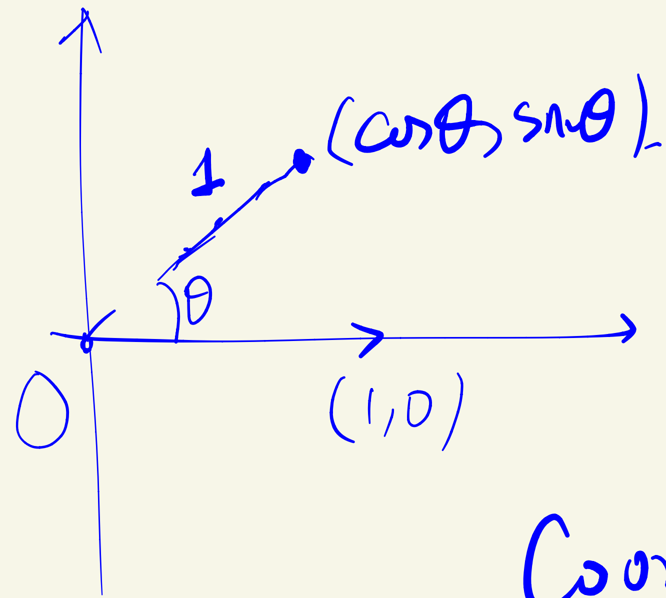


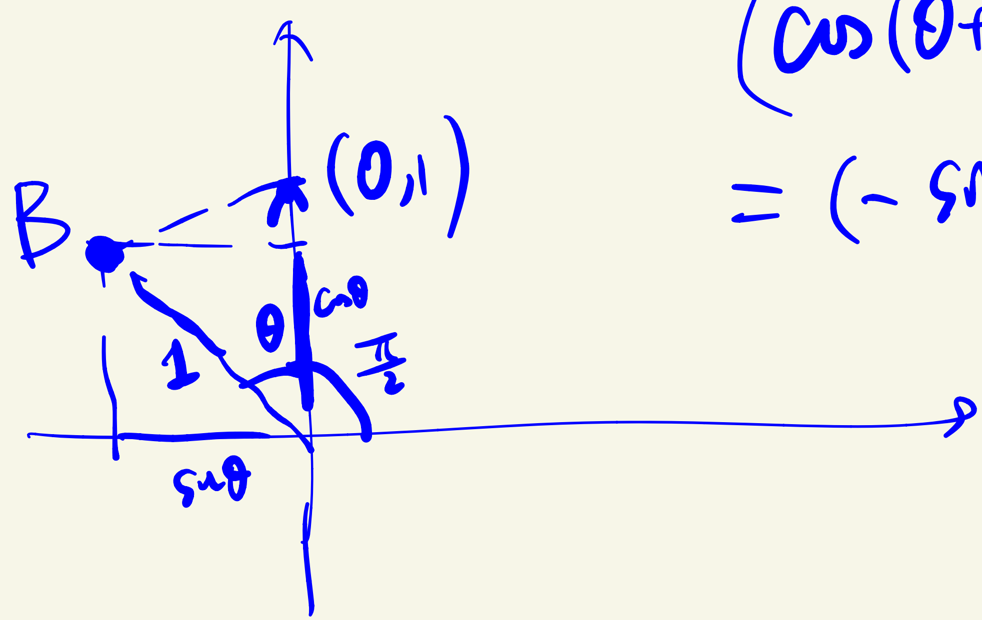
$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto T(x)$

$= x_1 T(e_1) + x_2 T(e_2)$

$= \begin{bmatrix} T(e_1), T(e_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$





Coordinate of B:

$$\left(\cos\left(\theta+\frac{\pi}{2}\right), \sin\left(\theta+\frac{\pi}{2}\right)\right)$$

$$= (-\sin\theta, \cos\theta).$$

Computing Linear Transformation

Proposition 20.1

If f is a linear transformation from from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$f(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i) = \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i), \quad \forall \alpha_i \in \mathbb{R}, \forall \mathbf{v}_i \in \mathbb{R}^n, i = 1, \ldots, k\,.$$

Q1: How to compute f? (If you know f is L.T.)

Algorithm 20.1

Step 1: Pick the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

Step 2: Compute $f(\mathbf{e}_i) \in \mathbb{R}^m, \forall i$.

Step 3 (?): For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x}) = \sum_i x_i f(\mathbf{e}_i)\,.$$

Enough to compute f(x).

But if want to write f(x) = Ax, need extra step.

Matrix of Linear Transformation

Step 3 (?): For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x}) = \sum_i x_i f(\mathbf{e}_i) \in \mathbb{R}^m .$$

Q2 How to write $f(x)$ as $f(x) = Ax$?

This formula computes $f(\mathbf{x})$ for any $\mathbf{x}$.

Can we write it as $f(\mathbf{x}) = A\mathbf{x}$ for some A?

$$f(x) = x_1 f(e_1) + \cdots + x_n f(e_n)$$

$$= \underbrace{\begin{bmatrix} f(e_1), & \cdots, & f(e_n) \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Computing Linear Transformation

Algorithm 20.1 (2nd variant) Answer Q2. What is A in $f(x) = Ax$?

Step 1: Pick the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

Step 2: Compute $f(\mathbf{e}_i) \in \mathbb{R}^m, \forall i$.

Step 3: Form a matrix $A = [f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)] \in \mathbb{R}^{m\times n}$

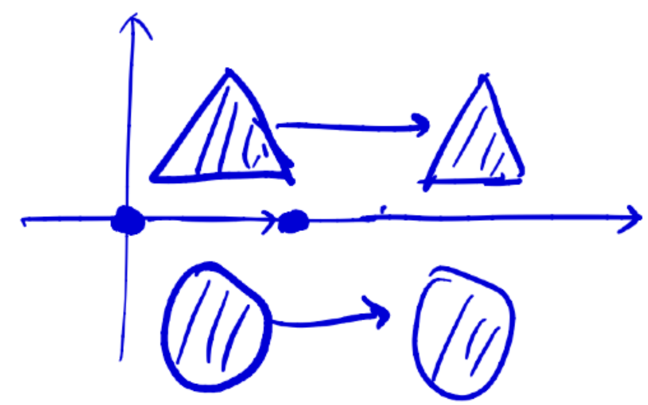
Conclusion: For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x}) = \sum_i x_i f(\mathbf{e}_i) = A\mathbf{x}\,.$$

A is the matrix of the linear transformation f.

For rotation, we've used Algorithm 20.1 to find A.

Analyzing Transformation: Eg1



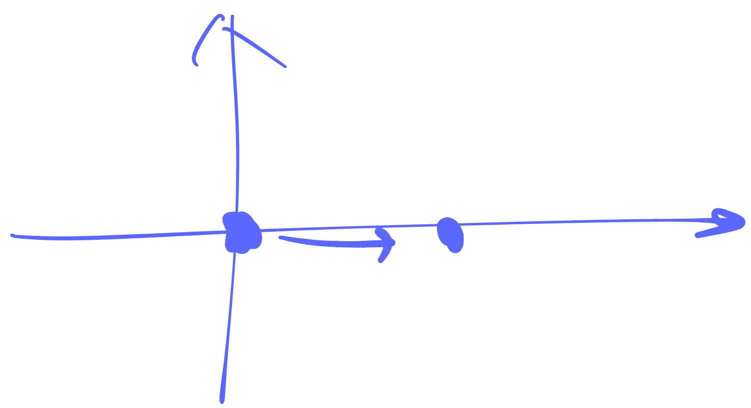
Eg (translation, 平移):

Suppose T(x_1, x_2) = $(x_1 + 1,\ x_2)$

Is it a linear transformation?

If so, find A such that T(x) = Ax

Intuition: Lines to lines; but origin NOT to origin .



(affine transformation, $f(x) = Ax+b$).

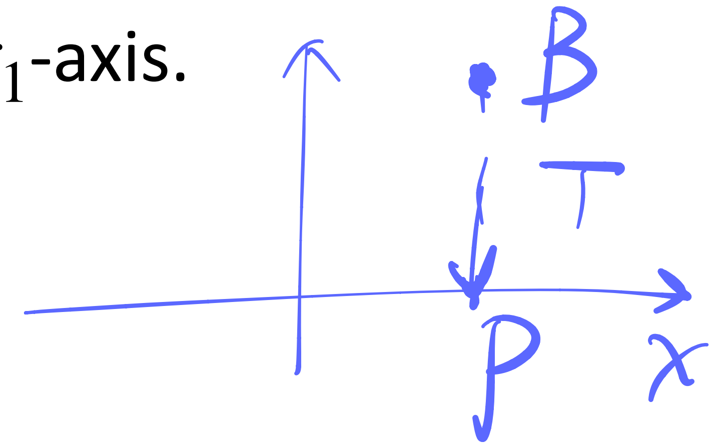
($Ax=b$; shifted linear space)

Analyzing Transformation: Eg2

Eg: Suppose T(x) is the projection of (x_1, x_2) onto x_1-axis.

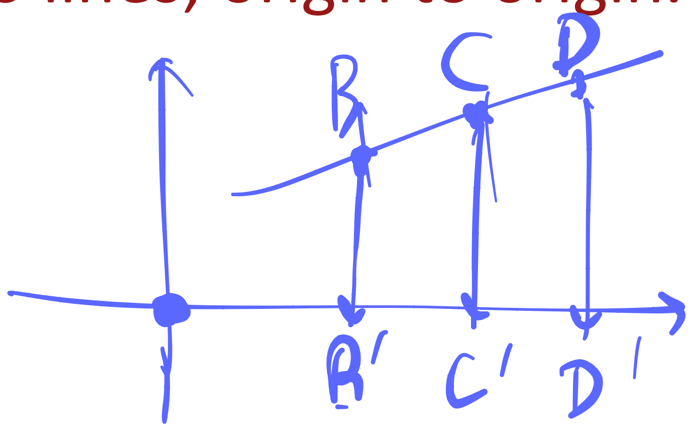
Is it a linear transformation?

If so, find A such that T(x) = Ax



Intuition: Lines to lines; origin to origin.

Method 1:



C is middle pt

middle pt C'

Method 2: T(x_1, x_2) = x_1

$$T(x_1, x_2) = \underbrace{\begin{bmatrix} 1, & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{so a L.T.}$$

Method 3:

$$T(x_1, x_2) = x_1 \cdot T(e_1) + x_2 \cdot T(e_2) = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1, & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analyzing Transformation: Eg3

Eg: Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as:

norm

$$L((x, y)^T) = \sqrt{x^2 + y^2}$$

Is it a linear transformation?

If so, find A such that $T(x) = Ax$

$L(\vec{v}) = \|\vec{v}\|$

L is not a linear transformation since

$$L(-\vec{v}) = L(-(x, y)^T) = L((-x, -y)^T)$$

$$= \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} \neq -L((x, y)^T)$$

$= \|\vec{v}\| = L(\vec{v}) \neq (-1) \cdot L(\vec{v})$

Eg: Projection onto Circles

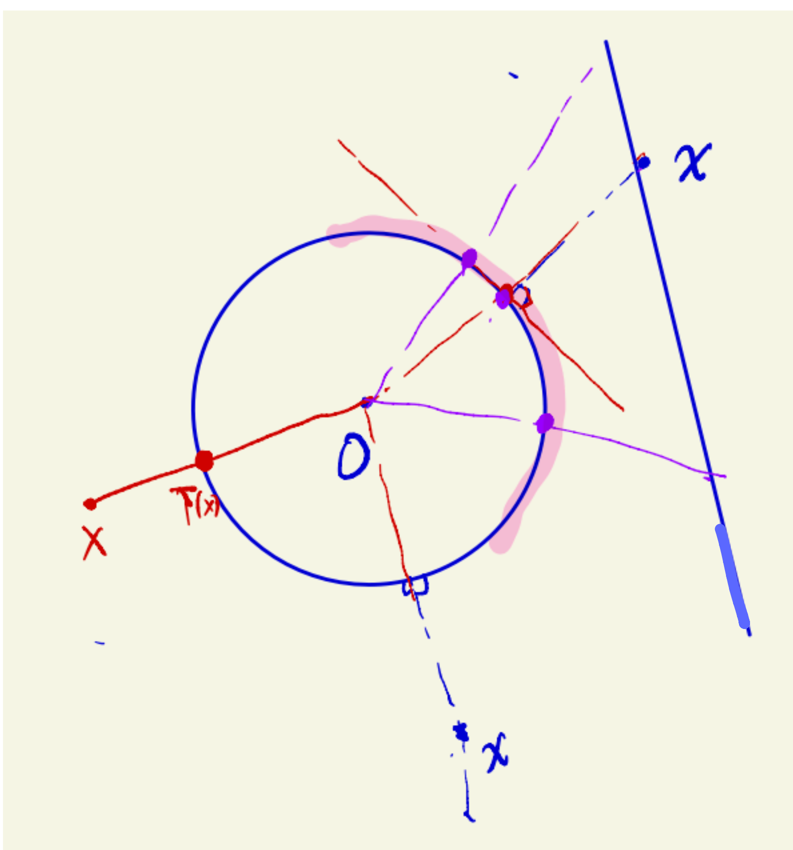
(Dropping apples to earth)

Transformation T: Given a circle C.

For any $X \in \mathbb{R}^2$, connect OX, and denote the intersection of OX and C as X'.

This X' is defined as $T(X)$.

Question: Is T a linear transformation?



<u>Answer</u>: No.

line $\xrightarrow{T}$ part of circle

NOT LT.

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Summary Today (Write Your Own)

One sentence summary:

We learned general linear transformation and how to “express” it.

Detailed summary: 1 Two definitions of Linear transformations.

2 **Deriving Expression of Linear transformation (Euclidean space)**

Find basis; Find the transformed vectors. Put in columns of a matrix.

—> Matrix of linear transformation.