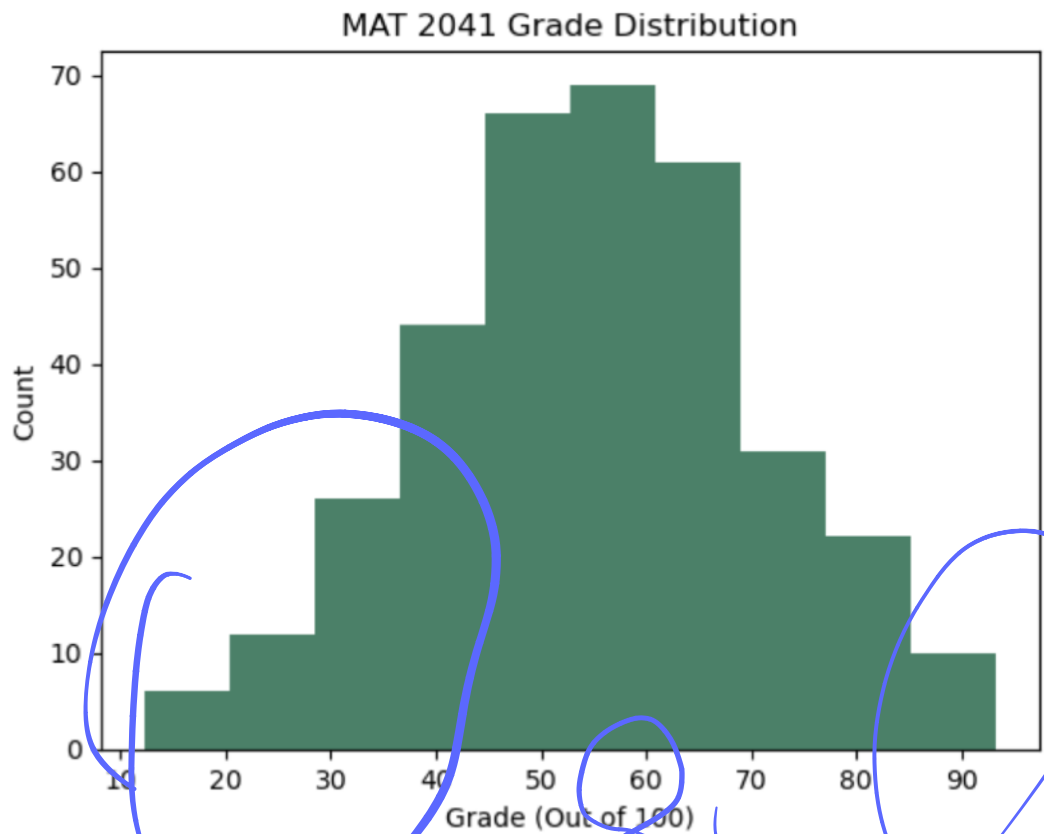
Lecture 21

Linear Transformation III

Instructor: Ruoyu Sun

香港中文大學(深圳) The Chinese University of Hong Kong, Shenzhen | 数据科学学院 School of Data Science

Logistics: Midterm Exam



final grade will be related to distribution

100-point system:

Average： 54.85/100

Medium： 55.56/100

Max: 93.3/100

Original (卷面):

Average: 49.3/90

Medium: 50/90

Max: 84/90

If you have difficulty, talk to me or the TA.

Today's Lecture: Outline

Main topic: Linear transformation II

1. View Matrix as Linear Transformation

2. Linear Transformation on General Linear Spaces

3. Matrix Representation of General Linear Transformation

Strang's book: Sec 8.1, 8.2

Today's Lecture: Learning Goals

After the lecture, you should be able to

1. Tell the relation of linear transformation and matrix
2. Verify linear transformation and compute it for general linear spaces
3. Derive the matrix representation of general linear transformation

Review

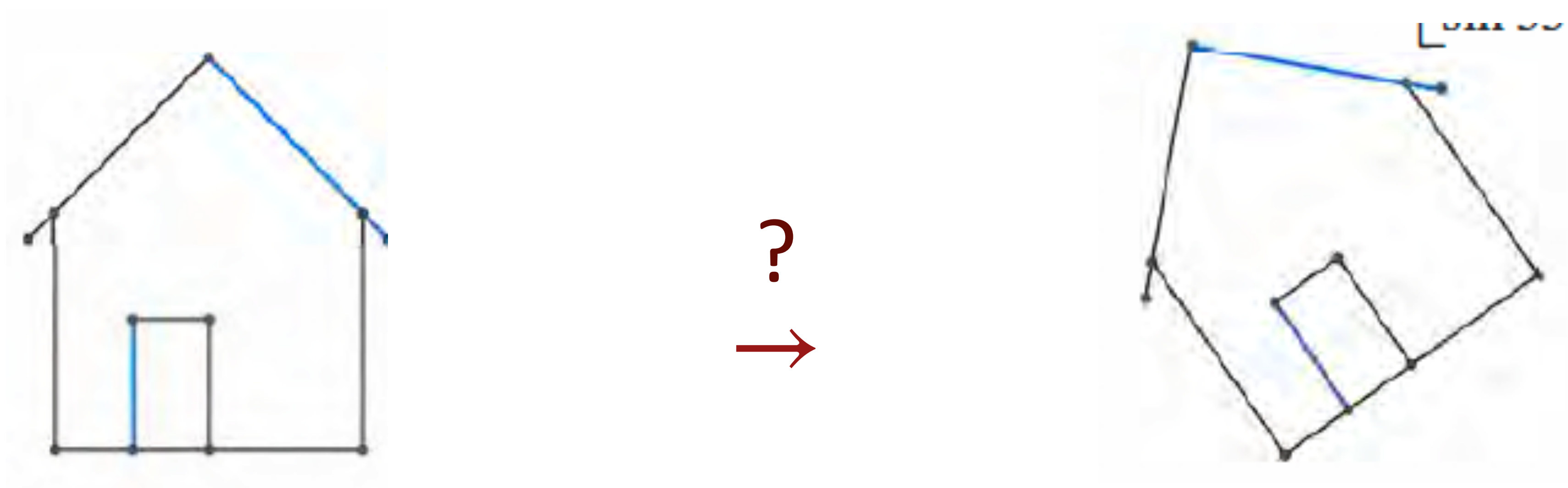
Rotate Photos

How to rotate photos on iPhone

Motivating Question

Yes, I know you can do it on the phones.

How do the phones accomplish the job?



Question 1: Is it a linear transformation?

Question 2: How do I derive its expression?

Linear Transformation in Euclidean Space: Two Definitions

Two equivalent definitions:

Definition 19.1 (matrix form):

Suppose $A \in \mathbb{R}^{m \times n}$ is a given real matrix.

$f(\mathbf{x}) = A\mathbf{x}$ is called a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 20.1 (alternative definition of LT):

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then we say f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Why Equivalent

Property 20.1 [Def 19.1 ==> Def 20.1]

Suppose f: $\mathbb{R}^n \to \mathbb{R}^m$ is defined as $f(\mathbf{x}) = A\mathbf{x}$, then

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n .$$

Theorem 20.1 [Def 20.1 ==> Def 19.1]

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then $f(\mathbf{x}) = A\mathbf{x}$ for some matrix A .

Computing Linear Transformation

f is L.T.

Algorithm 20.1

Step 1: Pick the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

Step 2: Compute $f(\mathbf{e}_i) \in \mathbb{R}^m, \forall i$.

output of the basis of input.

Step 3: Form a matrix $A = [f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)] \in \mathbb{R}^{m\times n}$

extend

Conclusion: For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x}) = \sum_i x_i f(\mathbf{e}_i) = A\mathbf{x}\,.$$

A is the matrix of the linear transformation f.

A

$$
\begin{tikzcd}
\text{Any L.T. } f \arrow[d] & \underline{f} \\
\text{find matrix } A \arrow[u, bend right, "?"'] &
\end{tikzcd}
$$

<u>Judgement:</u> For any matrix A, can find linear transformation whose matrix is A.

<u>Answer:</u> True. $\underline{f(x) = Ax}$

Part I View Matrix as Linear Transformation

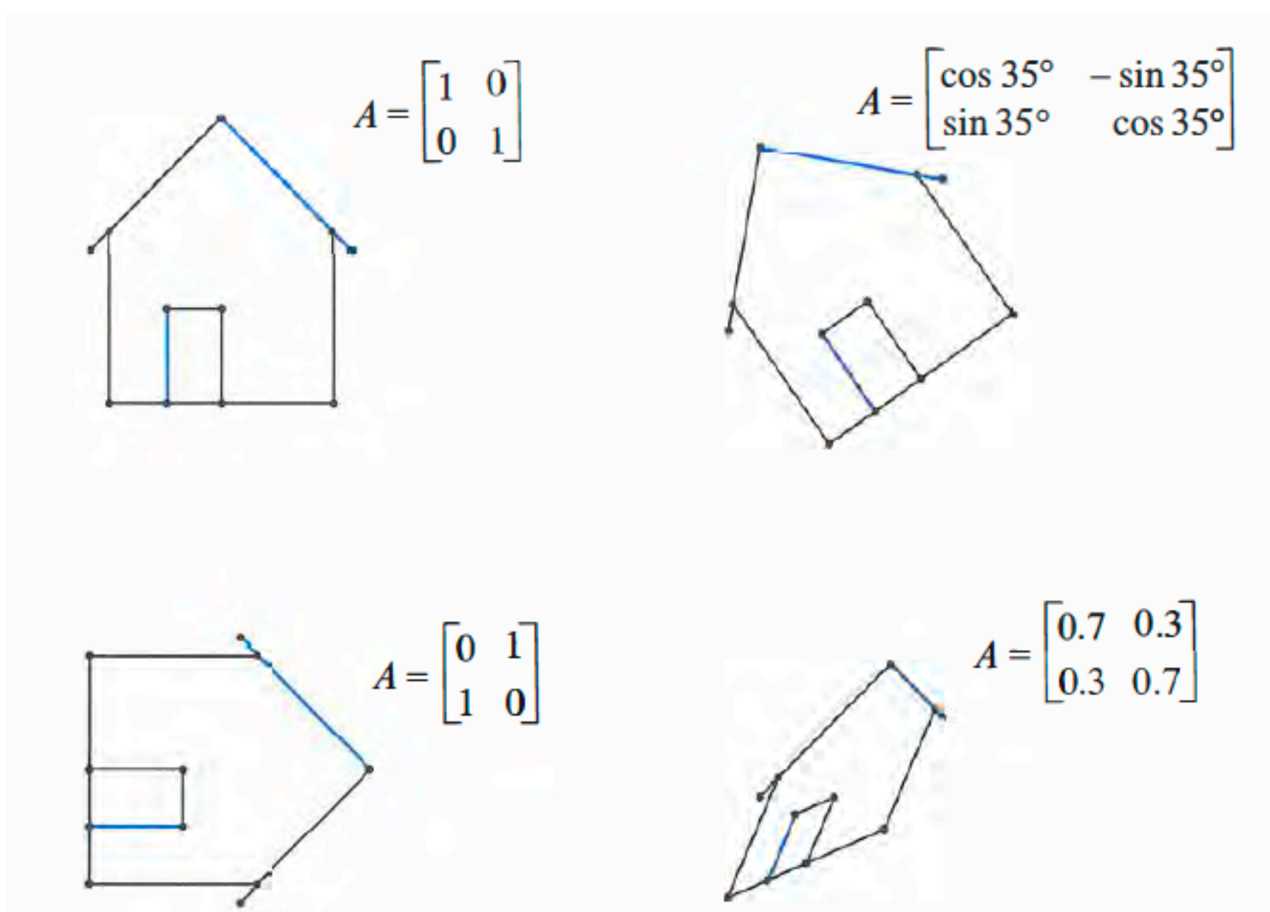
Important New View of Matrix

2D array

Previously: Matrix is a representation of data. (e.g. image)

New View: Matrix <—> a linear transformation.

Columns of the matrix <—> transformed basis vectors.



Transformation —> Matrix

$f(x) = Ax$

Matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = [\vec{a}_1, \vec{a}_2]$

Multiplying A: $e_1, e_2 \xrightarrow{A\cdot}$ columns of A.

$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ column 1

$A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ column 2

$$\overbrace{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{e_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot 0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$T(e_1)$ || column 1 of A

Any vector $\vec{x}$:

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 \xmapsto{A\cdot}$ Ax

$x_1 \cdot \text{colm } 1 + x_2 \cdot \text{colm } 2.$

$x_1 \cdot T(e_1) + x_2 \cdot T(e_2)$

Conclusion: Matrix A defines a linear transformation Ax

It maps $\vec{e}_1, \vec{e}_2$ to $\vec{a}_1, \vec{a}_2$ (columns of A). → explain

trivial

Matrix —> Transformation

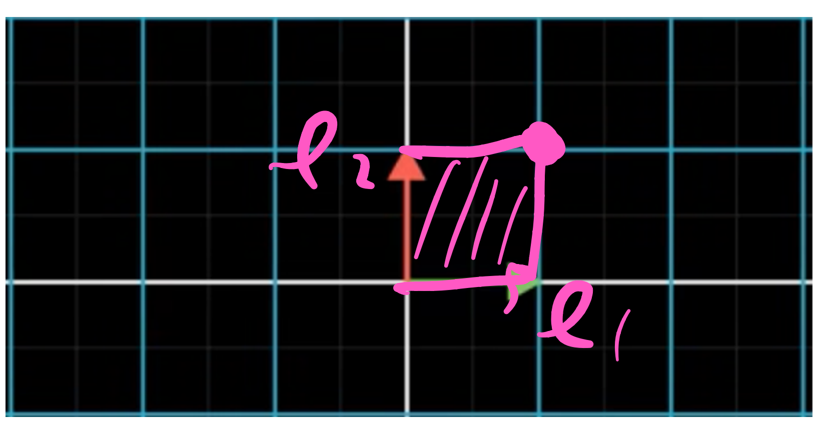
Matrix —> a linear transformation.

Columns of the matrix —> transformed basis vectors.

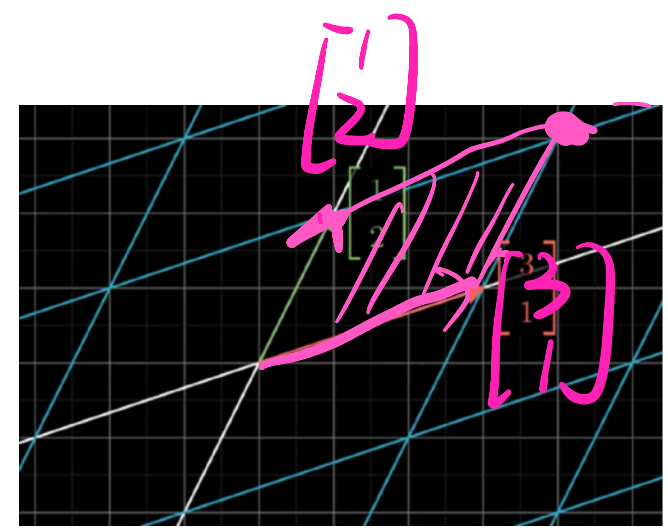
Matrix A: $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

basis

Linear transformation Ax:



$\longrightarrow$



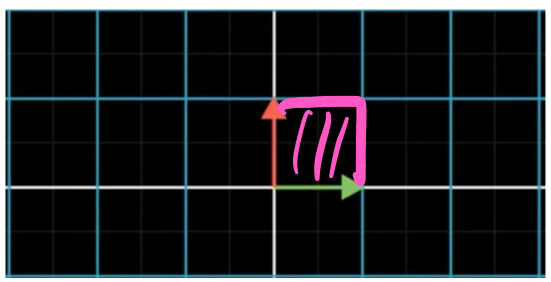
Input space:
"Square tile"
瓷砖

Output space:
"Parallelogram tile"

Determinant: Geometry Interpretation II

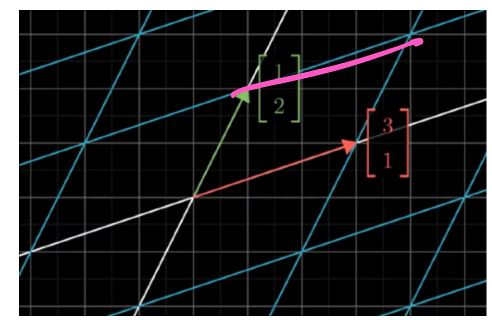
Matrix A: 

Linear transformation Ax:



Input space:
"Square tile"

area 1



Output space:
"Parallelogram tile"

area det(A)

Lec 18: Area of PARA($[a_1, a_2]$) is det(A).

Linear transformation perspective:

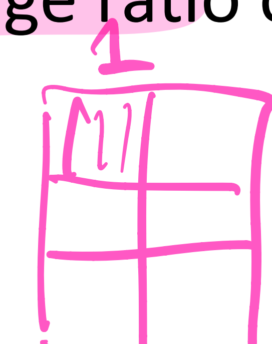
det(A) = area of PARA($[a_1, a_2]$)

= volume change ratio of linear transformation A

total area:

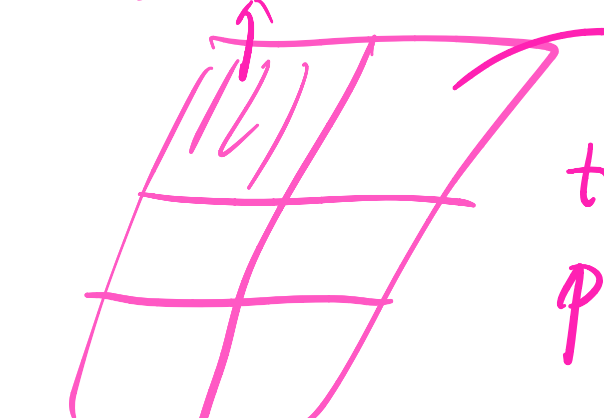
p

1



Volume change
by det(A) times

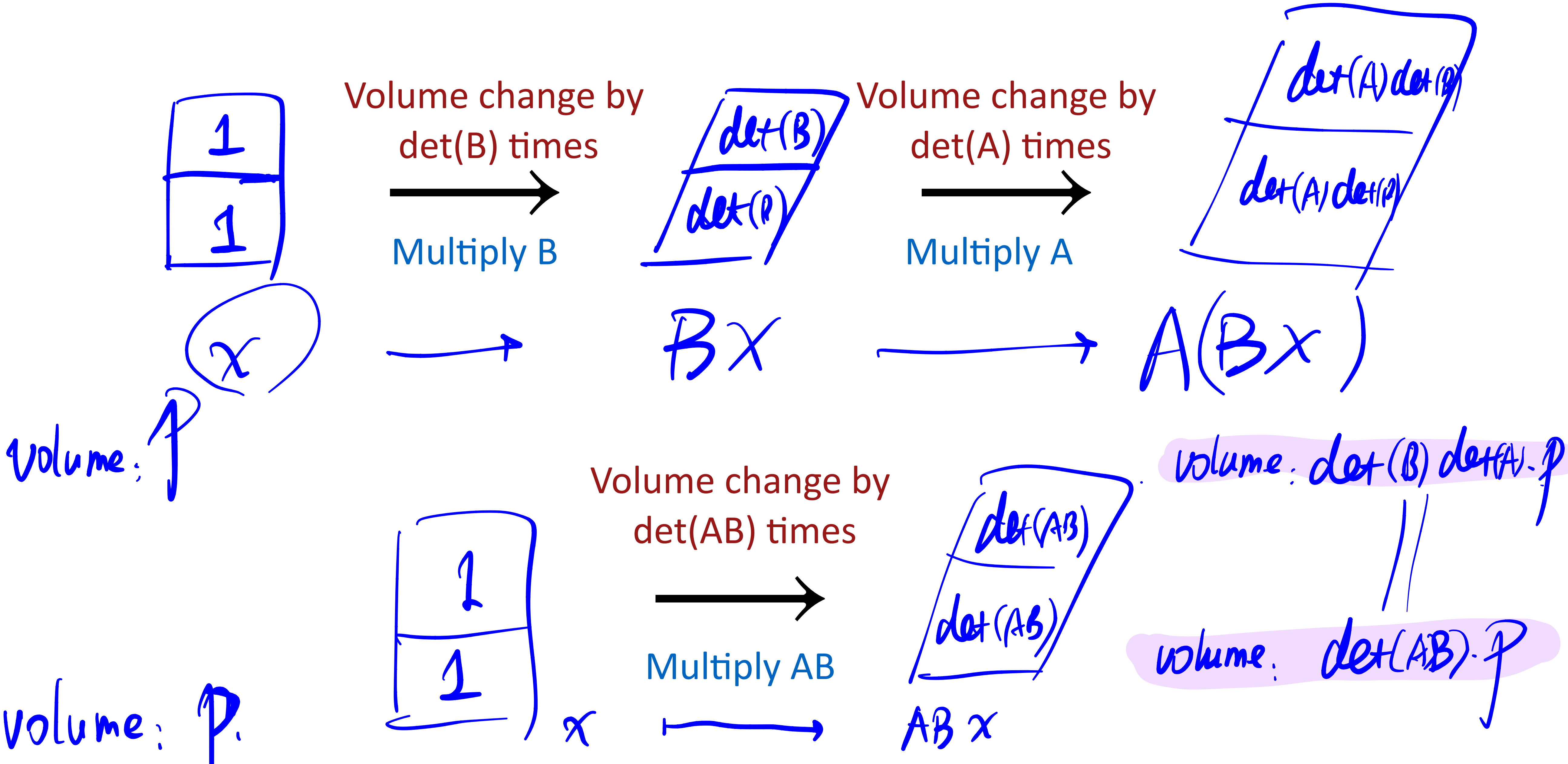
area det(A)



area det(A)

total area:
p det(A)

Determinant: Geometry Interpretation II



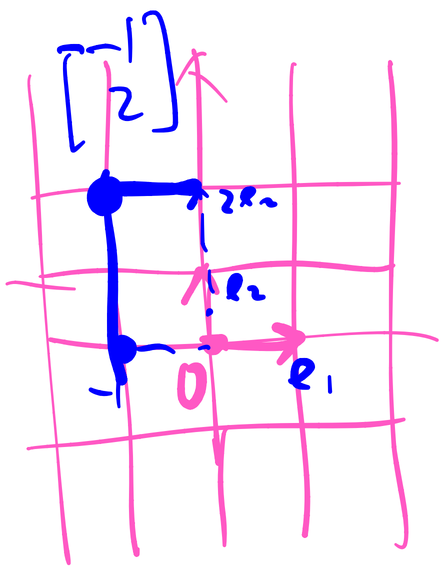
Geometrical proof of $\det(AB) = \det(A)\det(B)$

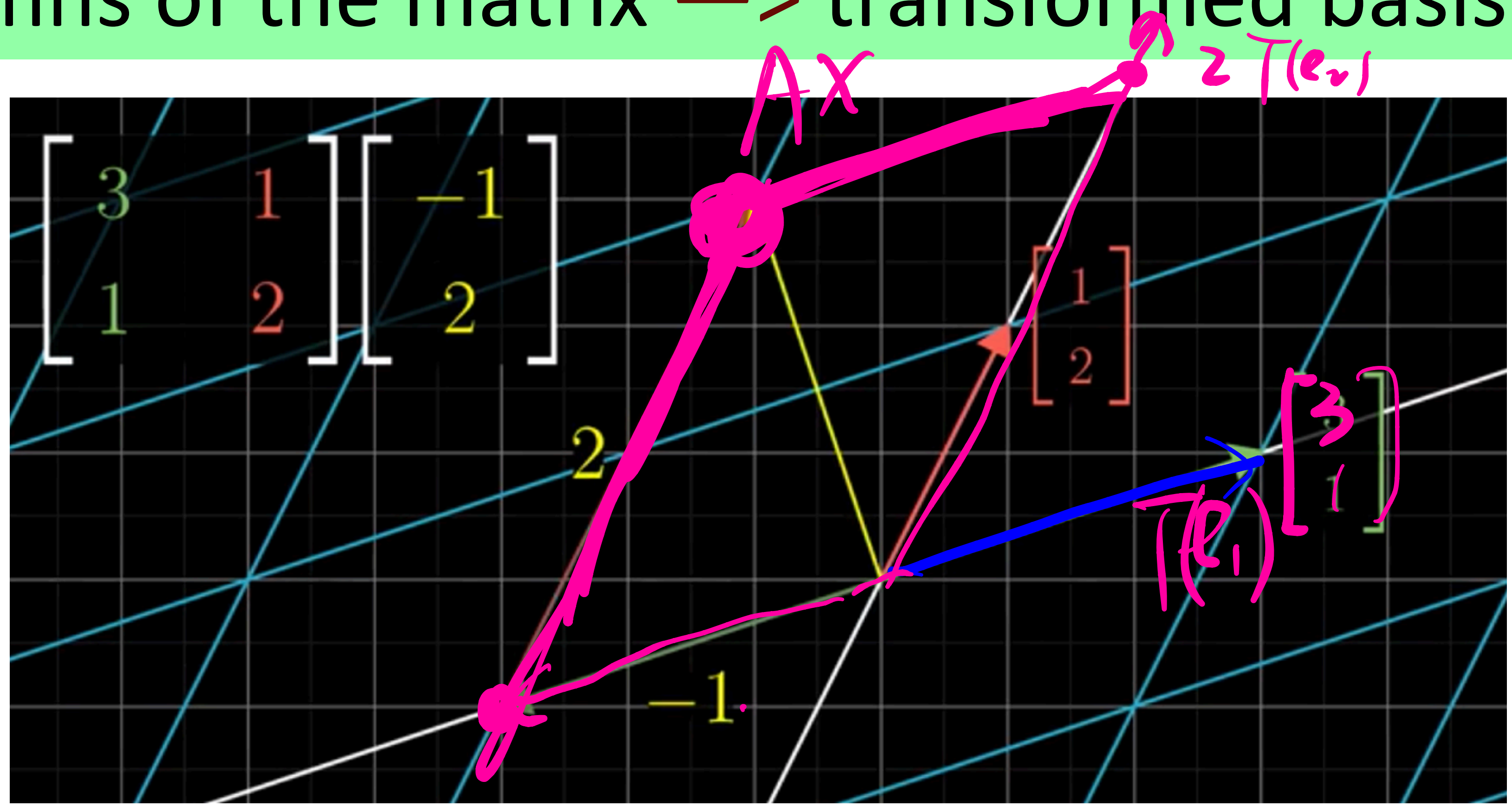
Matrix —> Transformation: coordinate (坐标)

Matrix —> a linear transformation.

Columns of the matrix —> transformed basis vectors.

$x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$





matrix A times vector (-1,2) gives a point with
coordinate (-1,2) in the new system where basis is columns of A

Matrix —> Transformation: Dependent Columns

Matrix —> a linear transformation.

Dependent columns of the matrix
—> transformed "basis vectors" are degenerate

$T(e_1) \,//\, T(e_2)$

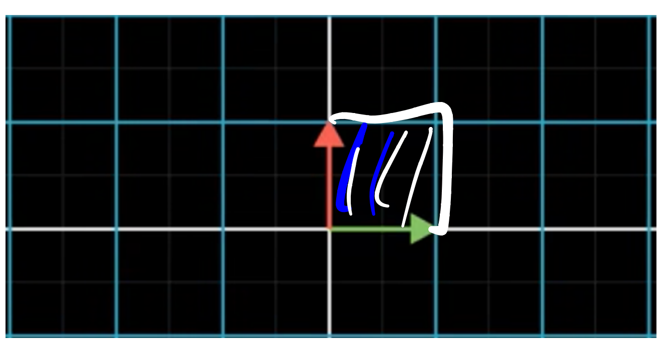
Matrix A: $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$

$A: \mathbb{R}^2 \to \mathbb{R}^2$
$\quad x \mapsto Ax$

The set $\{Ax : x \in \mathbb{R}^2\}$
$=$ Column space of A
$= \{\alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ -1 \end{bmatrix} : \alpha, \beta \in \mathbb{R}\} = \text{span}\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)$

$T(e_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad T(e_2) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Linear transformation Ax:



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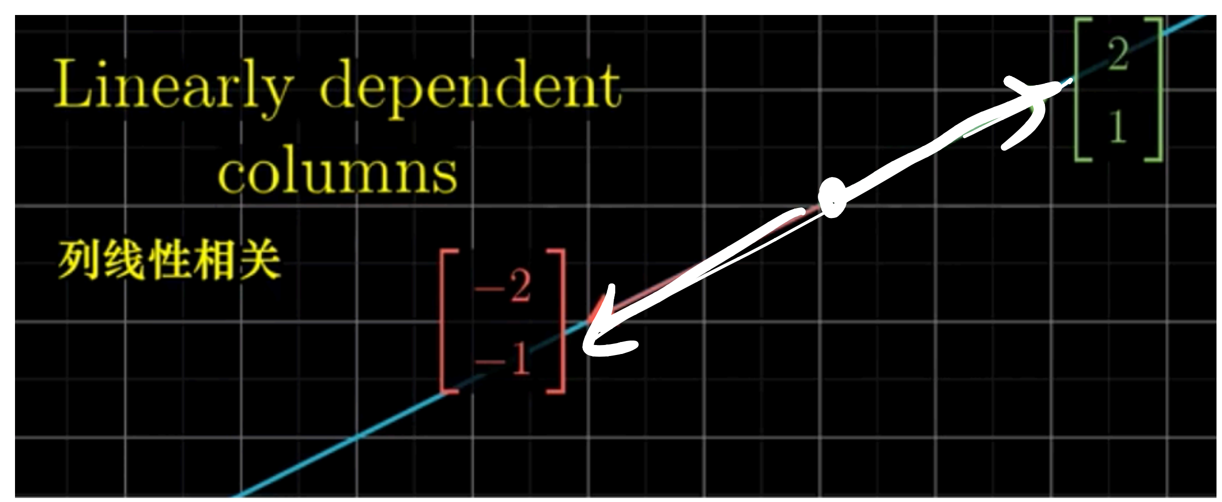


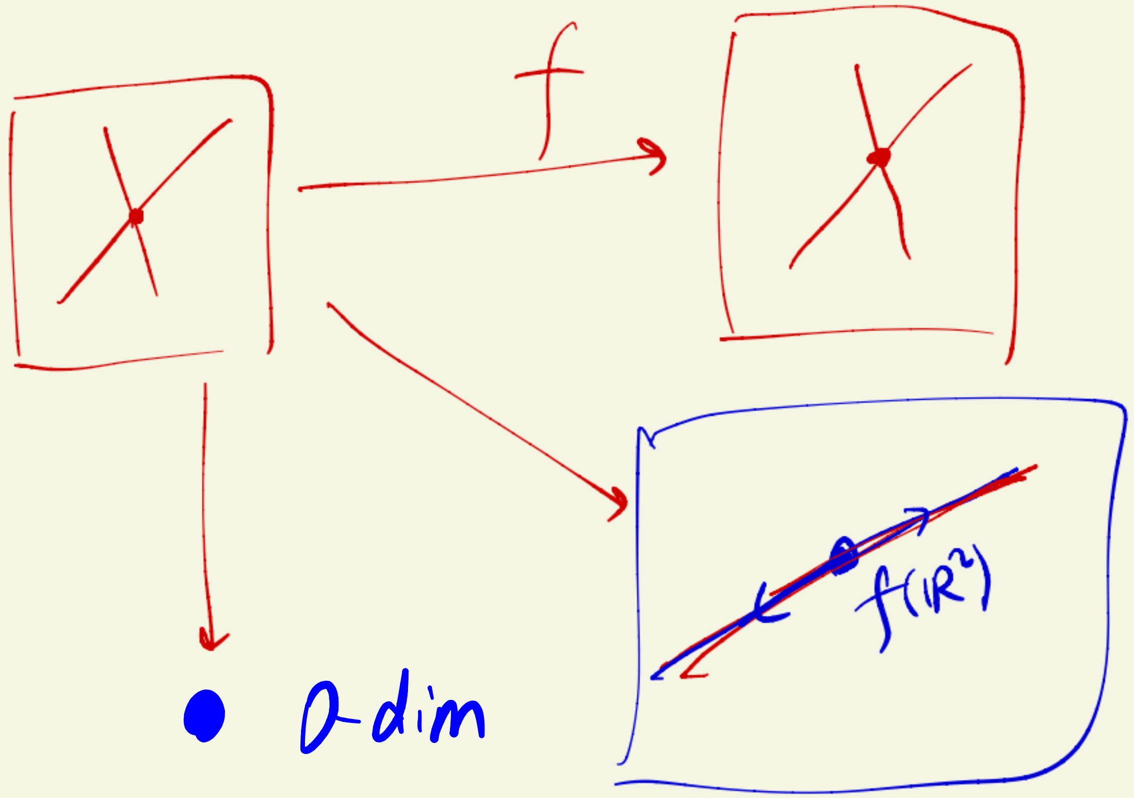
Three Cases of $f(\mathbb{R}^2)$ $\triangleq \{f(x);\ x \in \mathbb{R}^2\} = C(A)$.

Suppose $f(x) = Ax$.

column space of A

$\|$

$\{f(x):\ x \in \mathbb{R}^2\} \triangleq f(\mathbb{R}^2)$ can be $\mathbb{R}^2$, a line or $\{0\}$.



2D subspace in $\mathbb{R}^2$ ($= \mathbb{R}^2$)

1D subspace in $\mathbb{R}^2$ (a line)

e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ \forall x$.

<u>Remark.</u> $\{0\}$ is 0-dimensional space.

Exercise

Judgement: Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a standard basis of $\mathbb{R}^n$.

Suppose $f(\mathbf{x}) = A\mathbf{x}$, where $A \in \mathbb{R}^{m \times n}$.

Then $f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n) \in \mathbb{R}^{m \times 1}$ form a basis of $\mathbb{R}^{m \times 1}$.

False.

Counter-example: $f(x) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} x$, $x \in \mathbb{R}^{2 \times 1}$.

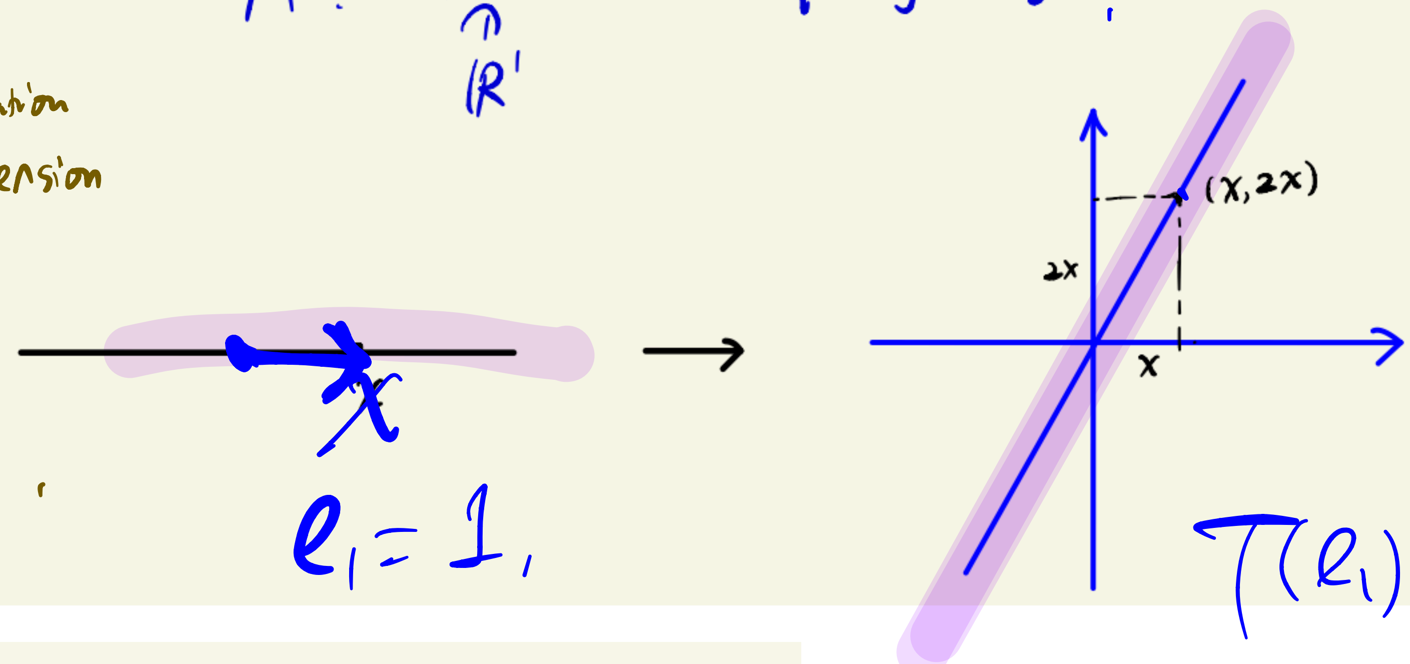
Then $\{f(e_1), f(e_2)\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

is not a basis of $\mathbb{R}^{2 \times 1}$.

Example: Mapping to Higher-Dim Space

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^{2\times 1} : \mathbb{R}^1 \to \mathbb{R}^2$

$A : \underset{\mathbb{R}^1}{\underset{\cap}{x}} \mapsto x \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$



$e_1 = 1,$

$T(e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A : e_1 = 1 \mapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix} = T(e_1)$

$A(\mathbb{R}^1) = \text{span}(Ae_1) = \text{subspace of } \mathbb{R}^2,$

$f(\mathbb{R}^1) \neq \mathbb{R}^2,$

$e_1 \to T(e_1)$

$\boxed{\text{Lesson from this page;}}$

The codomain of a linear transformation can have higher dimension than the dim of the domain

Note;

$f: \mathbb{R}^n \to \mathbb{R}^m,$

If $n < m,$

$f(\mathbb{R}^n)$ is a subspace of $\mathbb{R}^m$, NOT $\mathbb{R}^m$.

Wrap-Up: What We Learned So Far?

How to understand linear transformation?

Def: $\Rightarrow T(\underbrace{x_1 e_1 + x_2 e_2}_{x}) = x_1 T(e_1) + x_2 T(e_2)$ $(*)$

This definition $(*)$ has a few implications.

(1) Interpretation: If you know $T(e_1), T(e_2)$ then you should know $T(x)$, $\forall x$.

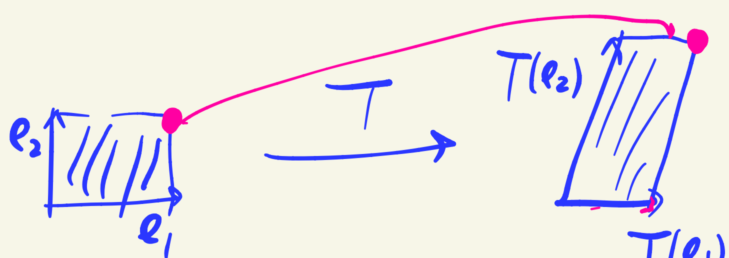
(2) coordinate $(x_1, x_2) \longrightarrow$ coordinate (x_1, x_2)

"basis" $(e_1, e_2) \longrightarrow (T(e_1), T(e_2))$ [Remark: $T(e_1), T(e_2)$ may be dependent. thus they are just spanning set of $\{Tx : x \in \mathbb{R}^2\}$.

This : coordinates stay, "basis" change $\rightarrow$ "tile change".

Today (Geometrical interpretation)

(3)



square tile — parallelogram tile

(4) One more corollary:
When $T(x) = Ax$,
we have: $T(e_1), T(e_2)$
are columns of A.

Part II Linear Transformation between Two Linear Spaces

Linear Transformation for Any Linear Space

Definition 21.1 (Linear Transformation)

Let V and W be two linear spaces.
A mapping $L : V \to W$ is called a **linear transformation** if
$L(\alpha v + \beta u) = \alpha L(v) + \beta L(u)$, for all $\alpha, \beta \in \mathbb{R}$, and $v, u \in V$.

V is the **domain** of the linear transformation

W is the **codomain** of the linear transformation

Remark: This is an extension of Def 20.1 from Euclidean spaces to any spaces.

Linear Transformation for Any Linear Space

Definition 21.1 (Linear Transformation)

Let V and W be two linear spaces.
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Remark:

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

for all $\alpha, \beta \in \mathbb{R}$, $v, w \in V$

$$\Longleftrightarrow$$

$$L(v + w) = L(v) + L(w)$$

$$L(\alpha v) = \alpha L(v)$$

for all $\alpha \in \mathbb{R}$, $v, w \in V$.

Simple Examples on Matrix Space

Eg: Suppose $X \in \mathbb{R}^{m \times n}$ is a matrix variable.

Then $f(X) = 2X$ is a linear transformation from $\mathbb{R}^{m \times n}$ to $\mathbb{R}^{m \times n}$.

$$f(\alpha X) \overset{?}{=} \alpha f(X)$$
$$2(\alpha X) = \alpha \cdot 2X$$

$$f(X+Y) \overset{?}{=} f(X) + f(Y)$$
$$2(X+Y) = 2X + 2Y$$

scalar dis...

Eg: Suppose $A \in \mathbb{R}^{k \times m}$ is fixed. $X \in \mathbb{R}^{m \times n}$ is a matrix variable.

Then $f(X) = AX$ is a linear transformation from $\mathbb{R}^{m \times n}$ to $\mathbb{R}^{k \times n}$.

$$f(\alpha X) \overset{?}{=} \alpha f(X)$$
$$A(\underbrace{\alpha}_{\text{scalar}} X) = \underbrace{\alpha}_{\text{scalar}} \cdot (AX)$$

$$f(X+Y) \overset{?}{=} f(X) + f(Y)$$
$$A(X+Y) = AX + AY$$

matrix distributive rule

Examples or Non-Examples on Matrix Space

Exercise: Is "inverse" a linear transformation?

Suppose $X \in \mathbb{R}^{n \times n}$ is a matrix variable.

$X: \mathbb{R}^{n\times n}$ $X^{-1}: \mathbb{R}^{n\times n}$.

Then $f(X) = X^{-1}$ ___is NOT___ a linear transformation.

Reason 1. X^{-1} NOT defined for every matrix X.

Reason 2. $(X+Y)^{-1} \neq X^{-1} + Y^{-1}$. $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$

Exercise: Is "transpose" a linear transformation?

Suppose $X \in \mathbb{R}^{m \times n}$ is a matrix variable. $\mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$

Then $f(X) = X^{\top}$ ___is___ a linear transformation.

$f(X+Y) = (X+Y)^T = X^T + Y^T = f(X) + f(Y)$

$f(\alpha X) = (\alpha \cdot X)^T = X^T \cdot \alpha^T = \underset{\substack{\downarrow \\ \text{scalar}}}{\alpha} \cdot X^T = \alpha f(X)$

Examples on Polynomial Space

Example

Let $T : P_3 \to \mathbb{R}^{2\times 2}$ be a mapping defined as:

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Is this L a linear transformation?

For any $\alpha \in \mathbb{R}$, $f, g \in P_3$,

Suppose $f(x) = \sum_{i=0}^{3} a_i x^i$, $g(x) = \sum_{i=0}^{3} b_i x^i$,

Then $(f+g)(x) = \sum_{i=0}^{3} (a_i + b_i) x^i$ → poly-addition

$L(f+g) = \begin{bmatrix} a_3+b_3 & a_2+b_2 \\ a_1+b_1 & a_0+b_0 \end{bmatrix} = \begin{bmatrix} a_3 & a_2 \\ a_1 & a_0 \end{bmatrix} + \begin{bmatrix} b_3 & b_2 \\ b_1 & b_0 \end{bmatrix} = L(f) + L(g)$

matrix addition

$L(\alpha f) = \begin{bmatrix} \alpha \cdot a_3 & \alpha \cdot a_2 \\ \alpha \cdot a_1 & \alpha \cdot a_0 \end{bmatrix} = \alpha \cdot \begin{bmatrix} a_3 & a_2 \\ a_1 & a_0 \end{bmatrix} = \alpha L(f)$

scalar times poly

scalar times matrix

Is Differentiation a Linear Transformation?

求导.

Eg: Is $T(u) = \dfrac{du}{dx}$ is a linear transformation?

$T(f) = f'$.

Reminder of notation: e.g. If $u(x) = x^2$, then $\dfrac{du}{dx} = 2x$.

Check:

$T(\alpha u) \stackrel{?}{=} \alpha T(u)$

$(\alpha \cdot u)' = \alpha \cdot u'$.

$T(u+v) = T(u) + T(v)$?

$(u+v)' = u' + v'$.

e.g. $(\sin x + e^x)'$

$= (\sin x)' + (e^x)'$

$= \cos x + e^x$.

$T(f) = f'$.

$T(x^2) = 2x$

$x^2 \mapsto 2x$

$T(\sin(x)) = \cos(x)$

$\sin x \mapsto \cos x$

$T(e^x) = e^x$

$e^x \mapsto e^x$

Domain Shall be Linear Space

Eg: Is $T(u) = \dfrac{du}{dx}$ is a linear transformation?

Definition 20.1 (Linear Transformation)

Let V and W be two linear spaces.

A mapping $L : V \to W$ is called a **linear transformation** i

Notice: T is not defined for all functions.

Need to specify a linear space s.t. **T** is defined!

Function Space

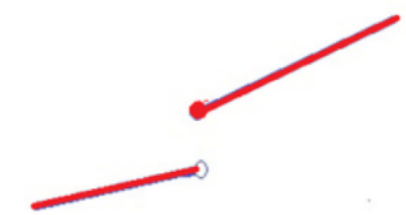
Continuous function space $C(\mathbb{R})$: The set of continuous functions form a linear space. (equipped with "function addition" and "scalar-function product")

Recall: Definition of two operations:

$(f+g)(x)$: value at $x = f(x)+g(x)$

$(\alpha f)(x)$: value at $x = \alpha \cdot f(x)$

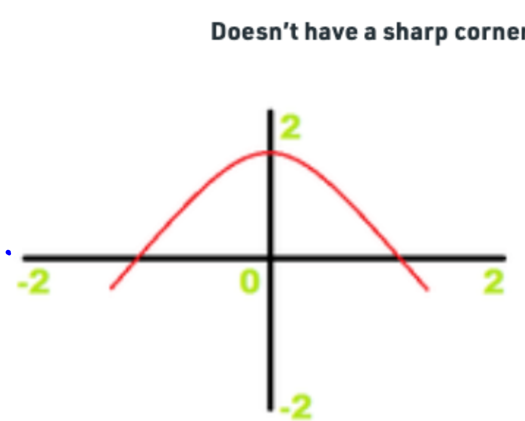
Need to show: These operations satisfy 8 axioms of Definition of linear space. [skipped here].

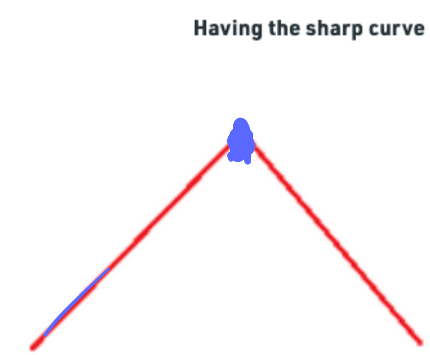


Continuously differentiable function space $C^1(\mathbb{R})$: a linear space.

set of functions whose derivative exists and is continuous function

$\triangleq \{ f: \mathbb{R} \to \mathbb{R} \mid f'(x)\ \exists$ and $f'(x)$ is a continuous function$\}$.





$f'(x) \nexists$

Need to verify the axioms.
Skip details here.

Differentiation is Linear Transformation

Claim: $T(u) = \dfrac{du}{dx}$ defined on $C^1(\mathbb{R})$ is a mapping from $C^1(\mathbb{R})$ to $C(\mathbb{R})$.

It is a linear transformation.

$T: C'(\mathbb{R}) \to C(\mathbb{R})$

$f(x) \mapsto \dfrac{df}{dx}$

Proof sketch: ① Input space $C'(\mathbb{R})$, output space are both linear spaces.

② T is well-defined on $C'(\mathbb{R})$, ③ $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$, $\forall \alpha, \beta \in \mathbb{R}$, $f, g \in C'(\mathbb{R})$.

Claim: $T(u) = \dfrac{du}{dx}$ defined on P_3 is a mapping from P_3 to P_2.

It is a linear transformation. [skip proof]

$T: P_3 \to P_2$

$ax^3+bx^2+cx+d \xmapsto{\text{differentiation}} 3ax^2+2bx+c.$

$$ax^3+bx^2+cx+d$$

$$\downarrow T$$

$$3ax^2+2bx+c.$$

Part III Matrix Representation of Linear Transformation for Any Linear Spaces

Recall: Two Definitions

Euclidean Space

Matrix

Definition 19.1 (matrix form):

Suppose $A \in \mathbb{R}^{m\times n}$ is a given real matrix.

$f(\mathbf{x}) = A\mathbf{x}$ is called a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

S.P.

Definition 20.1 (alternative definition of LT):

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$

Then we say f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

extend

General Space

Matrix ?

A.

Definition 21.2 (Linear Transformation)

Let V and W be two linear spaces.

A mapping $L : V \to W$ is called a **linear transformation** if

$L(\alpha v + \beta u) = \alpha L(v) + \beta L(u)$, for all $\alpha, \beta \in \mathbb{R}$, and $v, u \in V$..

Matrix Representation of Differentiation

$T(u) = \dfrac{du}{dx}$ defined on P_3 is a linear transformation from P_3 to P_2.

~~Is T related to a matrix?~~

Try to apply Algorithm 20.1?

FUNCTION

Algorithm 20.1

Step 1: Pick the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

Step 2: Compute $f(\mathbf{e}_i) \in \mathbb{R}^m, \forall i$.

Step 3: Form $A = [f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)] \in \mathbb{R}^{m \times n}$

Conclusion: For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x}) = \sum_i x_i f(\mathbf{e}_i) = A\mathbf{x}.$$

But how to form a matrix, if no vectors?

P_3: $\underset{e_1}{\overset{\|}{1}},\ \underset{e_2}{\overset{\|}{x}},\ \underset{e_3}{\overset{\|}{x^2}},\ \underset{e_4}{\overset{\|}{x^3}},$

$[T(e_1), T(e_2), T(e_3), T(e_4)]$

$= [1, 0, 2x, 3x^2]$.

Recall: Vector Representation of Linear Space

Proposition (Unique Representation via Basis)

$f+g$

$c_0(f) + c_0(g)$

Let $\mathscr{U} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of a linear space X.

Any element $x \in X$ can be **uniquely** represented as a linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. [discussed in a lecture before]

P_3 : basis $\{1, x, x^2, x^3\}$.

$$f(x) = \underset{b_0}{a_0} + a_1 x + a_2 x^2 + \underset{b_3}{a_3} x^3 = \overbrace{[1, x, x^2, x^3]}^{\text{functions}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$\downarrow$ coordinate

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{matrix} b_0 \\ \vdots \\ b_3 \end{matrix}.$$

Matrix Representation

$$X \longrightarrow Y$$

$$v_1 \mapsto T(v_1)$$

$$v_2 \mapsto T(v_2)$$

$$x = [v_1, v_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto y = T(x) =$$

Matrix Representation

$$X \longrightarrow Y$$ new basis of Y: u_1, u_2, u_3

$$v_1 \mapsto T(v_1) = [u_1, u_2, u_3]\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = [u_1, u_2, u_3]\ \vec{a_1}$$

$$v_2 \mapsto T(v_2) = [u_1, u_2, u_3]\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = [u_1, u_2, u_3]\cdot \vec{a_2}$$

previous view: spanning set $\{v_1, v_2\} \rightarrow \{T(v_1), T(v_2)\}$.

but $[T(v_1), T(v_2)]$ is NOT a matrix.

$$x = \underbrace{[v_1, v_2]}_{\text{old basis}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto y = T(x) = [T(v_1), T(v_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

express $T(v_i)$ by basis $\{u_1, u_2, u_3\}$.

$$= \left[(u_1, u_2, u_3)\vec{a_1},\ (u_1, u_2, u_3)\vec{a_2}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

split to get "matrix"

$$= \underbrace{(u_1, u_2, u_3)}_{\text{new basis}}\ [\vec{a_1}, \vec{a_2}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Relation to Part I

Last Page: | Part 1.

Any linear spaces:		Euclidean Spaces.	
$X \xrightarrow{T} Y$		$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$	
$\dim(X)=2$	$\dim(Y)=3$.	$X=\mathbb{R}^2$	$Y=\mathbb{R}^3$
Basis: $\{v_1, v_2\}$	$\{u_1, u_2, u_3\}$	$\{e_1, e_2\}$	$\{\eta_1, \eta_2, \eta_3\}$
$x \longmapsto y$		$x \longmapsto y$	

$$x = [v_1, v_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad\qquad x = \underbrace{[e_1, e_2]}_{I_2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\longmapsto y = [u_1, u_2, u_3][\vec{a}_1, \vec{a}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad\qquad \longmapsto y = \underbrace{[\eta_1, \eta_2, \eta_3]}_{I_3}[T(e_1), T(e_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [T(e_1), T(e_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Lesson about relation of last page & Part I:

If we pick $\{v_1, v_2\}$, $\{u_1, u_2, u_3\}$ as standard basis of $\mathbb{R}^2$ & $\mathbb{R}^3$, then we get the derivation of Part I.

For general linear spaces, we have extra $[v_1, v_2]$, $[u_1, u_2, u_3]$, to relate X, Y to Euclidean space.

Expression for General Linear Space

Computing T(u):

$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ **must transform to**

$T(u) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$

Suppose you know $T(v)$ for all vectors $v_1, \ldots, v_n$ in a basis

Then you know $T(u)$ for every vector u in the space.

Matrix representation of T:

T(v_j) is a combination $a_{j1}w_1 + \ldots + a_{jn}w_n$ of the output basis for W.

The matrix $(a_{ij})_{m\times n}$ is the matrix representation of T.

Key rule: The jth column of A is found by applying T to the jth basis vector v_j

Matrix Representation of T: Algorithm

Algorithm 21.1 (for general linear space)

Step 1: Pick a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of input space V.

Step 2: Compute $f(\mathbf{v}_i) \in W, \forall i$.

Step 3: Find vectors.

Step 3.1: Pick a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of output space W.

Step 3.2: Express $f(\mathbf{v}_i) = \sum_{j=1}^{m} a_{ij}\mathbf{u}_j = [\mathbf{u}_1, \ldots, \mathbf{u}_n] \begin{bmatrix} a_{i1} \\ \cdots \\ a_{im} \end{bmatrix}$

Step 4: Form $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$

Remark: Once A is computed, it can be used to compute $f(x)$ for any x.

Step 5: Compute $y = f(x)$, for any $x \in V$.

Step 5.1 Express x as $x = \sum_{i=1}^{n} x_i v_i$.

Step 5.2 Compute $\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Step 5.3 Compute $y = \sum_{j=1}^{m} y_j u_j$.

[Explain Step 5].

Diagram: View L.T. as Operating on Coordinates

If we know "matrix A", how to compute $f(x)$?

Basis: $\{v_1, \ldots, v_n\}$ X $\longrightarrow$ Y Basis: $\{u_1, u_2, \ldots, u_m\}$.

Original Space

$$x = \sum_{i=1}^{n} x_i v_i \qquad\qquad y = \sum_{j=1}^{m} y_j \cdot u_j$$

omit basis, coordinate

expression of y based on "coordinate"

Euclidean space.

Coordinate of x Coordinate of $y = T(x)$

$$(x_1, \ldots, x_n) \xrightarrow{\text{multiply } A} (y_1, \ldots, y_n) = A\,x$$

Matrix Representation of Differentiation

<u>Problem 1.</u> Consider $T(u) = \frac{du}{dx} : P_2 \to P_1$.

Find the matrix of T under the basis of input space $\{1, x, x^2\}$

and the basis of the output space $\{1, x\}$.

<u>Method 1</u>. (intuitive way).

$$T: \ ax^2 + bx + c \longmapsto 2ax + b$$

Under the two bases, the coordinates are:

$$\vec{w} = [c, b, a]^T \longmapsto \vec{z} = [b, 2a]^T \overset{\text{want}}{=\!=} A\vec{w}.$$

Since $\begin{bmatrix} b \\ 2a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix}$.

Thus $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Method 2 (Use Algorithm 21.1)

We compute the representation of $T(v_1), T(v_2)$ under the basis of P_1: $\{1, x\}$

$v_1 = 1$, $T(v_1) = 0 = [1, x]\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, under basis of P_1, coordinate $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_2 = x$, $T(v_2) = 1 = [1, x]\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, under basis of P_1, coordinate $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_3 = x^2$, $T(v_3) = 2x = [1, x]\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, under basis of P_1, coordinate $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Combine three vectors, get matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

How to Use "Matrix" Form?

Problem 2. Compute $T(u) = \frac{du}{dx}$ for any $u \in P_2$.

Method 1 (classical way)

For any $ax^2+bx+c \in P_2$,

from calculus rule, $T(ax^2+bx+c) = 2ax+b$.

Method 2. (matrix form)

For any $f(x) = a+bx+cx^2 \in P_2$,

under basis $\{1, x, x^2\}$, its coordinate is $[a,b,c]^T$.

Compute $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix}$.

Thus output $T(f) = [1, x]\begin{bmatrix} b \\ 2c \end{bmatrix} = b+2cx$. $\square$

Remark: Method 2 is more complicated than Method 1, thus not necessary for this specific example.
It serves as an illustration of Algorithm 21.1, and help you understand better what "differentiation" is from a different angle.

Matrix Representation of Differentiation

$T(u) = \frac{du}{dx}$ defined on P_2 is a linear transformation from P_2 to P_1.

Is T related to a matrix?

Answer:

Under the basis of input space $\{1, x, x^2\}$

and the basis of the output space $\{1, x\}$,

$$\boxed{\boldsymbol{A} = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{2} \end{bmatrix} = \text{matrix form of the derivative } T = \frac{d}{dx}.}$$

Input u
$a + bx + cx^2$

Multiplication $Au = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix}$

Output $\frac{du}{dx} = b + 2cx.$

Exercise.

Problem. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is fixed.

Consider $f(X) = AX - XA$, where $X \in \mathbb{R}^{2\times 2}$.

Prove f is a linear transformation,

and find the matrix form of f.

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Summary Today (Write Your Own)

One sentence summary:

We learned general linear transformation and how to “express” it.

Order

Detailed summary:

Linear transformation view of matrix.

—Columns of a matrix are $T(e_i)$’s

—det(A) = volume change ratio of the transformation

Linear transformation between two linear spaces.

—Define by superposition property.

—Differentiation is a linear transformation.

—**Matrix representation of general linear transformation**:

Step 1: Find the transformed input basis elements;

Step 2: Find their vector representations under output basis;

Step 3: Combine these vectors to get a matrix.