Lecture 21

Eigenvalues and Eigenvectors

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Roadmap of Semester

Segment 1

Solving linear systems (Lec 3-15);

Solving least squares problem (Lect 16,17)

Segment 2

[Lec 18-21]: Two relatively independent parts:

—Determinant. [Important tool!]

—Linear transformation. [Advanced math perspective of matrix]

They are not directly related to solving system/problem,
but are fundamental and useful tools.

Segment 3

Next: Lec 22-27: Eigenvalues and related.

Tentative schedule:

—Eigenvalues. Lec 22-24

—Singular values. Lec 25-26

—Quadratic forms. Lec 27.

Today's Lecture: Outline

Main topic: eigenvalues and eigenvectors

1. Definitions of eigenvalues and eigenvectors
2. How to compute: Characteristic Polynomial
3. How many eigenvalues?

Strang's book: Sec 6.1

Today's Lecture: Learning Goals

After the lecture, you should be able to

1. Tell the definitions of eigenvalues and eigenvectors

2. Write characteristic polynomial, and compute eigenvalues and eigenvectors for small matrices

3. Tell how many eigenvalues does a matrix have

Review

Review

Why do we learn linear transformations?

Rotation as an example

Linear Transformation in Euclidean Space

Two equivalent definitions:

Definition 19.1 (matrix form):

Suppose $A \in \mathbb{R}^{m \times n}$ is a given real matrix.

$f(\mathbf{x}) = A\mathbf{x}$ is called a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 19.2 (alternative definition of LT):

If a mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

Then we say f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Linear Transformation: General Space

Definition 20.1 (Linear Transformation)

Let V and W be two linear spaces.

A mapping $L : V \to W$ is called a **linear transformation** if

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w) \qquad (20.*)$$

for all $\alpha, \beta \in \mathbb{R}$, and $v \in V, w \in W$.

V is the **domain** of the linear transformation

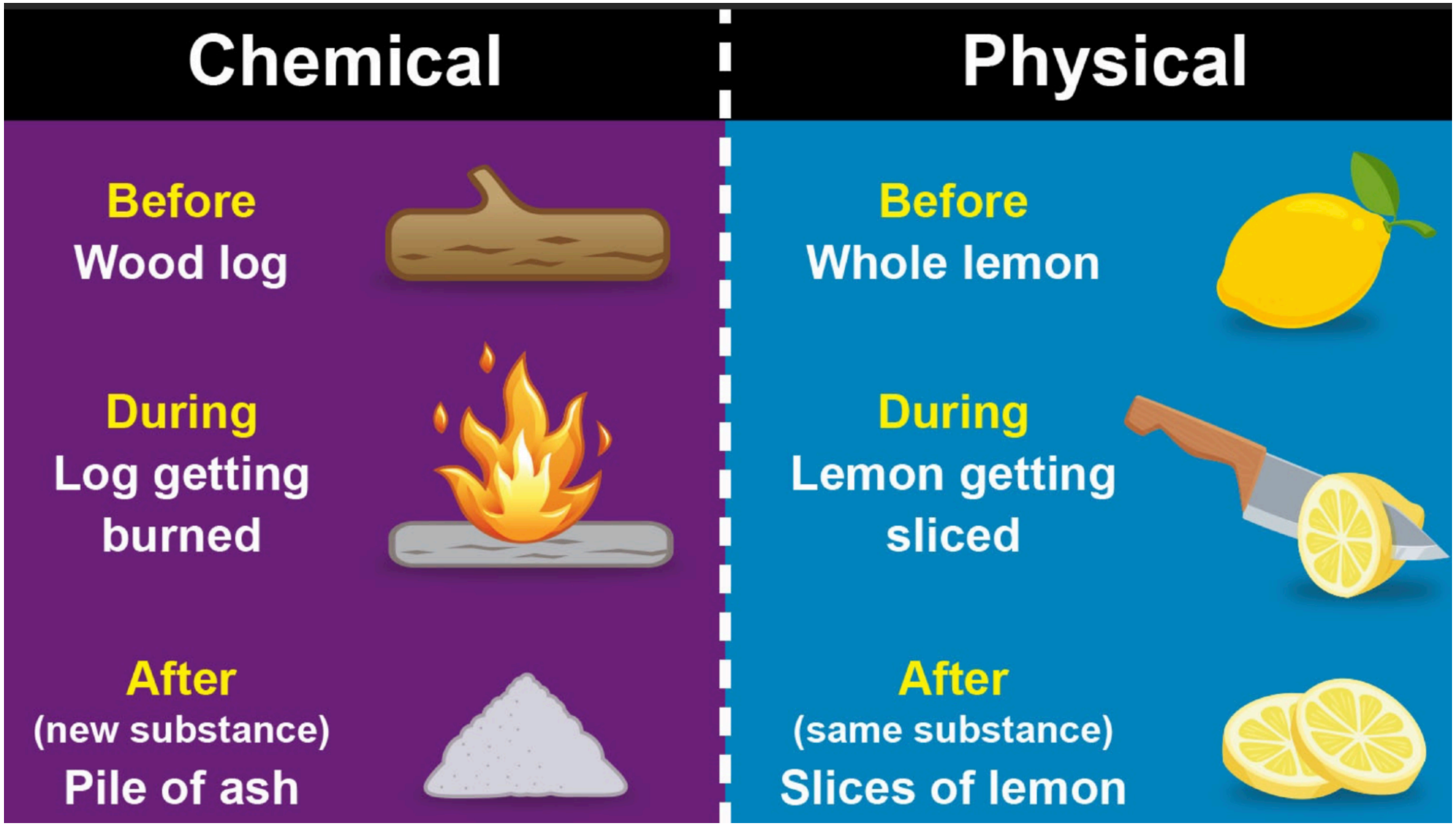
W is the **codomain** of the linear transformation

$$(20.*) \iff \begin{aligned} L(v + w) &= L(v) + L(w) \\ L(\alpha v) &= \alpha L(v) \end{aligned}$$

for all $\alpha \in \mathbb{R}, v \in V, w \in W$

Part I Motivation and Definitions of Eigenvalues and Eigenvectors

“Unchanged” During Change 变化中的不变



Chemical change:
Molecules change
[atoms are unchanged]

Physical change:
Molecules are unchanged

“Unchanged” During Change 变化中的不变

People love to understand “unchanged” parts during change.

Example in Science: physical change

Example of Chinese Ancient History (one popular theory):
“中国两千年之政，秦政也”。 (谭嗣同)
Two Thousand Years of policies, just Qin Dynamsty’s policy.
“外儒内法”。
These policies are “Confucious principles, Law Implementations”.

Why?

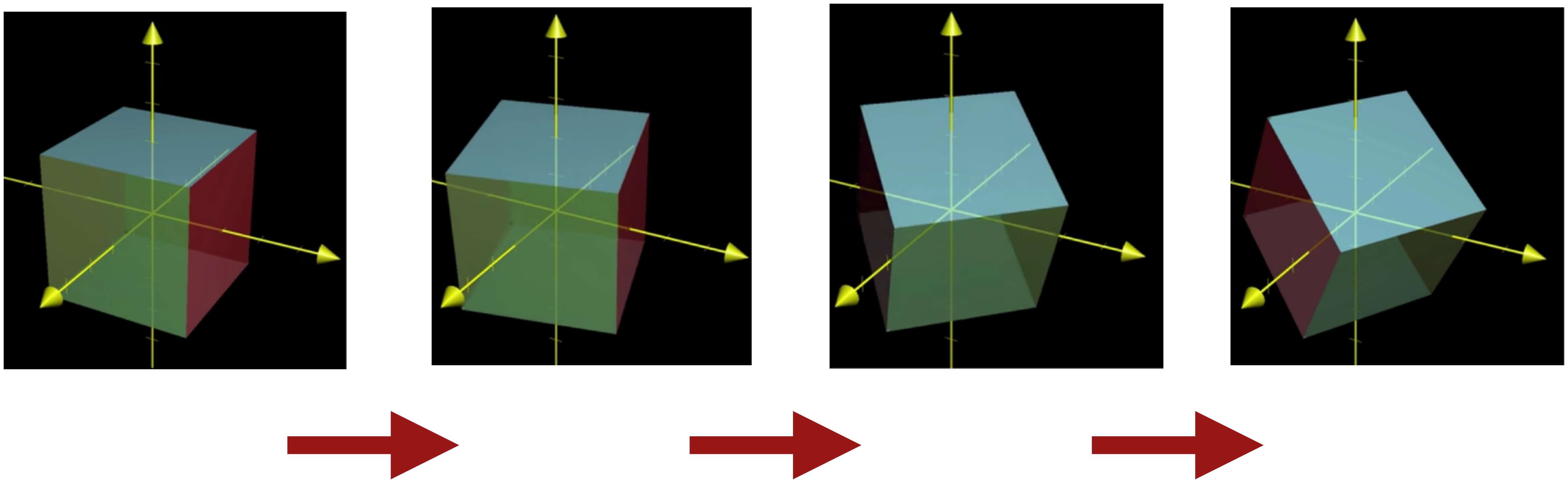
“Unchanged” During Change 变化中的不变

People love to understand “unchanged” parts during change.

Why?

Better understanding the essence of things.

Example: 3D Rotation

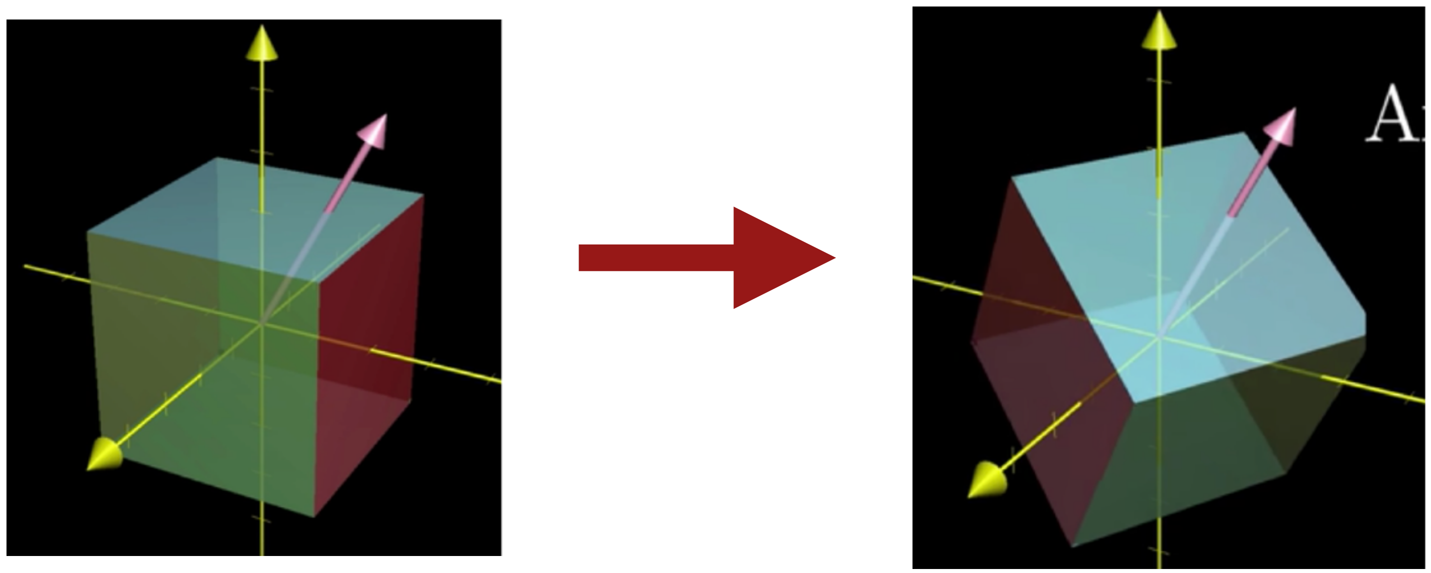


Matrix of linear transformation

$$\begin{bmatrix} \cos(\theta)\cos(\phi) & -\sin(\phi) & \cos(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) & \cos(\theta) & \sin(\theta)\sin(\phi) \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

“Unchanged” During Change: Axis of Rotation

Axis of rotation: Unchanged direction during rotation



Rotate 30° around $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

绕（2, 3, 1）旋转30°

V.S.

$$\begin{bmatrix} \cos(\theta)\cos(\phi) & -\sin(\phi) & \cos(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) & \cos(\theta) & \sin(\theta)\sin(\phi) \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

Find x and Ax on the same line?

Consider a 3×3 3D-rotation matrix A, find axis of rotation is just finding x s.t.

____

Question: Given some $A \in \mathbb{R}^{n \times n}$, can we find an $x \in \mathbb{R}^n$ such that x and Ax are on the same line?

Mathematically: $Ax = \lambda x$ for some $\lambda \in \mathbb{R}$

Eigenvalues and Eigenvectors

Definition 21.1 (Eigenvalues and Eigenvectors)

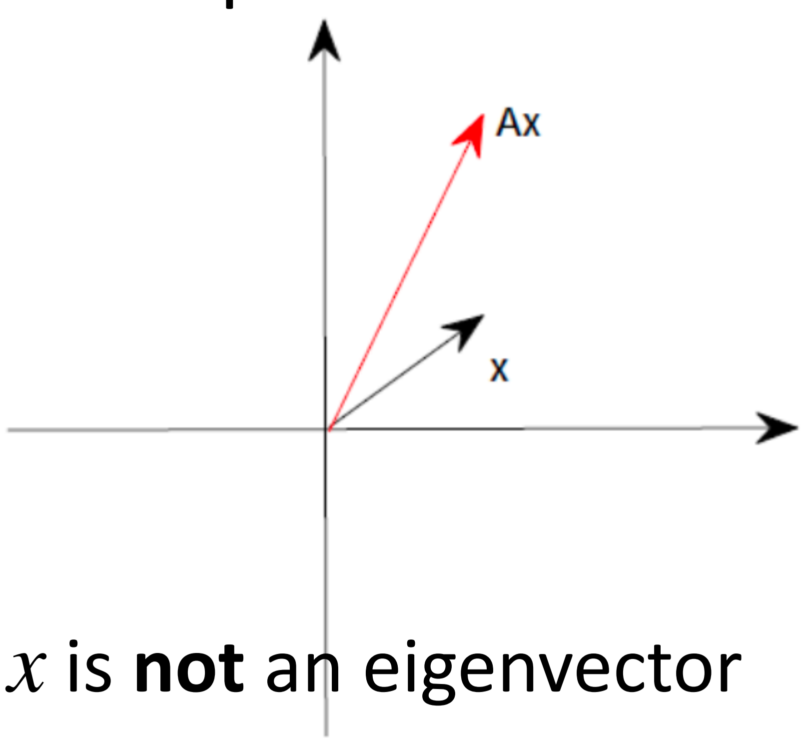
Let $A \in \mathbb{C}^{n \times n}$ be a square matrix.

If there exists a scalar λ ($\in \mathbb{R}$ or $\mathbb{C}$) and a **nonzero** vector x such that

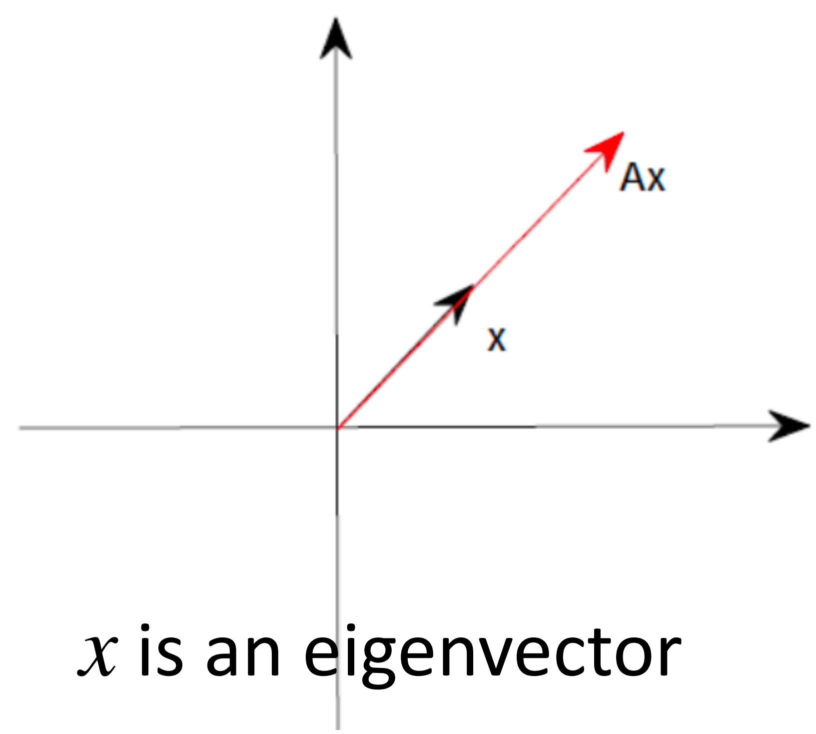
$$Ax = \lambda x,$$

then λ is called a (real or complex) **eigenvalue** and x is called an **eigenvector** with respect to (or associated with; corresponding to) λ.

$\mathbb{C}$:= set of complex numbers



x is **not** an eigenvector



x is an eigenvector

Example 1

Consider a 2×2 matrix A= $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x .$$

Thus _______ is _______________

_______ is _______________

Example 2

Consider a 2×2 matrix A= $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$Ax = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = x.$$

Thus ________ is ________________

________ is ________________

What do Eigenvalues Represent?

Recall:

Zooming transformation, what are the eigenvalues?

Eigenvalues: Measures the "size" of the change.

What are Eigenvalues and Eigenvectors?

Eigenvectors: “Anchor” vectors.

—Directions do not change.

(More later)

Eigenvalues: Measures the “size” of the change.

(More later)

Part II Characteristic Polynomial

How to Compute Them?

We showed how to verify eigenvalues and eigenvectors.

How to compute eigenvalues and eigenvectors?

$$Ax = \lambda x$$

Question to think: Is this a linear system of equations?

How to Compute Them?

We showed how to verify eigenvalues and eigenvectors.

How to compute eigenvalues and eigenvectors?

$$Ax = \lambda x$$

Question to think: Is this a linear system of equations?

No!!! Both λ and x are unknowns.

How to Simplify It?

Luckily, we can simplify this equation.

Consider

$$Ax = \lambda x \text{ for some } \lambda \in \mathbb{R} \text{ and nonzero } x \in \mathbb{R}^n$$

$\iff$ $Ax - \lambda x = 0$ for some $\lambda \in \mathbb{R}$ and nonzero $x \in \mathbb{R}^n$

$\iff$ $(A - \lambda I)x = 0$ for some $\lambda \in \mathbb{R}$ and **nonzero** $x \in \mathbb{R}^n$

$\iff$ $A - \lambda I$ is **not invertible** for some $\lambda \in \mathbb{R}$ **(Theorem 15.2)**

$\iff$ $\det(A - \lambda I) = 0$ for some $\lambda \in \mathbb{R}$ **(Thm 15.2++ in Lec 18)**

Example 1

Example

Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

Eigenvectors?

$\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ has two eigenvalues 4 and -3.

After finding eigenvalues, how to find eigenvectors?

Algorithms to Find Eigenvalues and Eigenvectors

Algorithm 21.1
(Finding all eigenvalues and eigenvectors of A)

Step 1: [Compute eigenvalues]

Solve $det(\lambda I_n - A) = 0$ to get roots $\lambda_1, \ldots, \lambda_p$, where $p \leq n$.

Step 2: [Compute eigenvectors]

For each λ_i, find $N(\lambda_i I_n - A)$, the eigenspace w.r.t. λ_i ,

i.e. any nonzero vector in $N(\lambda_i I_n - A)$ is an eigenvector w.r.t. λ_i .

What is this Function?

What is $\det(A - \lambda I)$ as a function of λ?

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}\right)$$

is a polynomial of λ!

Characteristic Polynomial

Definition 21.2 (Characteristic Polynomial)

Let $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) be a square matrix and λ be a variable.

Then,

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}\right)$$

is called the characteristic polynomial of A, and

$$p_A(\lambda) = 0$$

is called the characteristic equation.

Example 2

Example 2

Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

Example 2

Example 2

Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

Then characteristic equation is

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = -\lambda(\lambda-1)^2 = 0$$

$p_A(\lambda)$ is a polynomial with degree 3.

Example 2

Example 2

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = -\lambda(\lambda-1)^2 = 0$$

Then $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$

$$A - 0I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace w.r.t. 0 is $\mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

Example 2

Example 2

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = -\lambda(\lambda-1)^2 = 0$$

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace w.r.t. 1 is $\mathrm{Span}\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$.

Eigenvalues and Eigenvectors of Matrix Transpose

Proposition 21.1

Let $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) be a square matrix.

If λ is an eigenvalue of A, then λ is also an eigenvalue of $A^\top$.

Proof: Hint: consider $\det(A - \lambda I) = 0$

Part III How Many Eigenvalues?

How Many Eigenvalues?

Judgement: An $n \times n$ matrix always has n eigenvalues.

No right or wrong answer, without further specification.

How many eigenvalues <==>

____

How Many Roots Do Polynomials Have?

Linear equation: $ax + b = 0$ has _____ root.
(Single-variable!)

Quadratic equation: $ax^2 + bx + c = 0.$

e.g. $x^2 = 0$ has ________ roots   or 2 roots counting multiplicity

e.g. $x^2 - 1 = 0$ has ______ roots

e.g. $x^2 + 1 = 0$ has _______ roots   or 2 complex roots

e.g. by root formula, there are ________________________ roots.

Example 3: Rotation Matrix

Example 3

Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 4

Example 4

Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Example 4

Example 4

Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

The characteristic equation is

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 + 4 = 0.$$

Thus $\lambda = 1 \pm 2i$. When $\lambda = 1 + 2i$, then

$$A - (1+2i)I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

The eigenvector w.r.t. $1 + 2i$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Example 4

Example 4

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 + 4 = 0.$$

When $\lambda = 1 - 2i$,

$$A - (1-2i)I = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

The eigenvector w.r.t. $1 - 2i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

Reading: High-Order Polynomials Have ____ Roots?

Cubic equation: $ax^3 + bx^2 + cx + d = 0$ has ____ roots.

Quartic equation has ____ roots.

Root formulas of 3rd and 4th order equations:
found in 16th century.

≥ 5 order: Was a long-time mystery.
Theorem: No solutions in radicals.
[Ruffini'1799, Abel'1824, Galois'1830]

Spectacular
Galois theory!

Root formulas: compute roots.
What about existence?

How Many Eigenvalues Does a Matrix A Have?

Theorem 20.1 (Fundamental Theorem in Algebra)

Every polynomial of degree n with complex coefficients has **exactly n complex roots** (counting with multiplicity).

Remark: This is the d'Alembert–Gauss theorem in algebra

This theorem implies that $p_A(\lambda)$, as a **polynomial of degree** n, has n complex roots, i.e., n **eigenvalues** (counting with multiplicity).

Remark: The eigenvalues may not be distinct!

Applications of Eigenvalues and Eigenvectors

[Control] Stability in linear control theory

[Optimization and learning] Gradient descent as a Krylov subspace method

[Image processing] Image compression

[Computer science] PageRank (PCA and SVD)

[Statistics] Limit states of Markov chains

[Physics] Solutions of linear PDEs

[Physics] Cascading failure analysis

We will talk more on the data science application **Next Week**!

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Summary Today (Instructor)

One sentence summary:

We learned eigenvalues and eigenvectors.

Detailed summary:

One **motivation**: Better characterize linear transformation.

Definition.

—$Ax = \lambda x$ for nonzero x, then λ is an eigenvalue, x is an eigenvector w.r.t. λ .

—Eigenspace: Space spanned by all eigenvectors w.r.t. λ.

Computation.

—det($\lambda I_n - A$) =0, characteristic polynomial.

—Find eigenvalues: Solve this polynomial equation;

—Find eigenvectors: solve equation $Ax = \lambda x$ for fixed λ .

How many eigenvalues?

—Degree-n polynomial, always n complex eigenvalues (counting multiplicity)