# Lecture 23

## *Eigenvalue II: Spectral Decomposition*

谱分解

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# Today’s Lecture: Outline

Main topic: Spectral Theorem

1. Eigenbasis

2. Spectral Theorem

# Today's Lecture: Learning Goals

After the lecture, you should be able to

1. Provide an example where eigenvectors do not form a basis

2. Describe the spectral decomposition and related properties

3. Explain the main proof steps of spectral decomposition

# Review

# Eigenvalues and Eigenvectors

**Definition 21.1** (Eigenvalues and Eigenvectors)

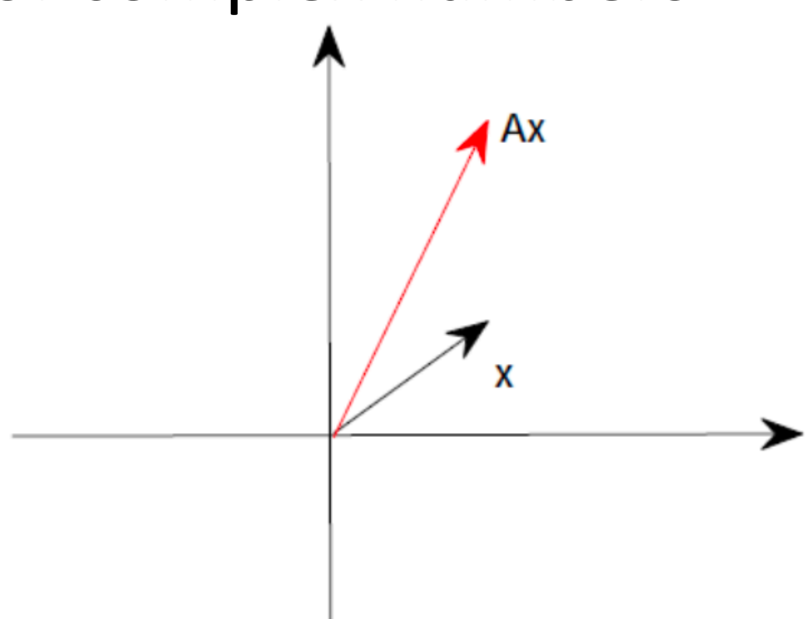
Let $A \in \mathbb{C}^{n \times n}$ be a square matrix.

If there exists a scalar $\lambda$ ( $\in \mathbb{R}$ or $\mathbb{C}$) and a **nonzero** vector $x$ such that

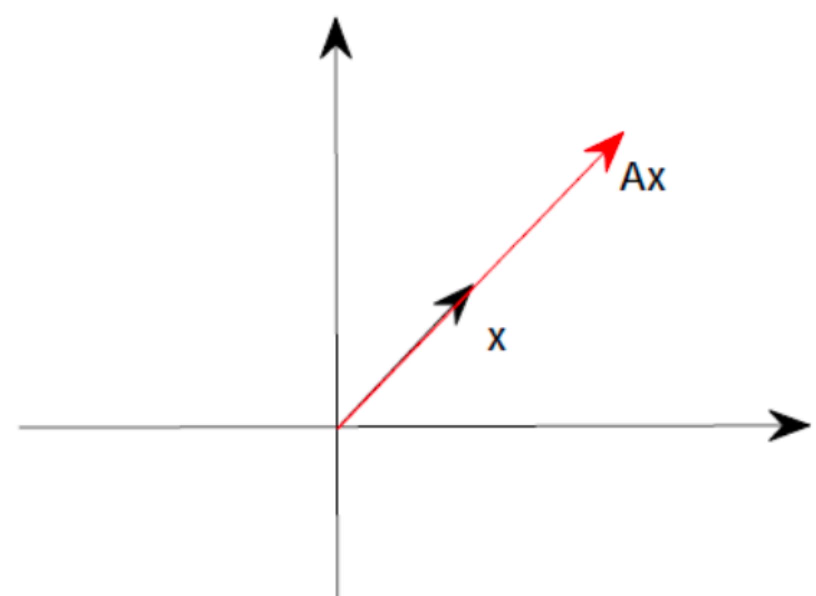
$$Ax = \lambda x,$$

then $\lambda$ is called a (real or complex) **eigenvalue** and $x$ is called an **eigenvector** with respect to (or associated with; corresponding to) $\lambda$.

$\mathbb{C}$:= set of complex numbers




$x$ is **not** an eigenvector




$x$ is an eigenvector

# General Procedure to Find Eigenvalues/Eigenvectors

**Step 1:** Solve $\det(A - \lambda I) = 0$ and get $n$ roots $\lambda_1, \ldots, \lambda_n$

Solving single-var polynomial.

deg $n$

**Step 2:** For each $\lambda_i$, find the eigenspace $\mathrm{Null}(\lambda_i I - A)$

Solving up to $n$ linear systems.

$A x_i = \lambda x_i$

Any nonzero vector in $\mathrm{Null}(\lambda_i I - A)$ is an eigenvector corresponding to $\lambda_i$

$$\mathrm{Null}(\lambda_i I - A) = \left\{ \begin{array}{l} \text{eigenvectors of } A \text{ with} \\ \text{respect to the eigenvalue } \lambda_i \end{array} \right\} \bigcup \{0\}$$

# Eigenvalues and Eigenvectors

$\det(\lambda I - A)$

Characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ is a degree-$n$ polynomial with coefficient of $\lambda^n$ being $(-1)^n$.

**Fact:**

Any $n \times n$ (no matter real or complex) matrix $A$ has exactly $n$ **complex eigenvalues** (counting multiplicity).

Here, multiplicity of $\lambda_j$ is the power $k$ of the term $(\lambda - \lambda_j)$ in the decomposition of the characteristic polynomial $p_A(\lambda) = (-1)^n \Pi_{j=1}^n (\lambda - \lambda_j)^{k_j}$.

$ax^2 + bx + c = a(x - x_1)(x - x_2)$

$x_1 = x_2 \quad = a(x - x_1)^2$

# Part 1  Eigenbasis

# Multiset [多重集]

~~$\{a, a, b\}$~~ ~~$\{1, 1, 2\}$~~

**Multiset** A multiset is a modification of the concept of a set that, unlike a set, allows for multiple instances for each of it.

Can use $\#\{a_1, a_2, \ldots\}$ to denote a multiset; though some ppl just use $\{a_1, a_2, \ldots\}$

$\#\{1, 1, 2\}$.

**Multiplicity:** If an element appears $k$ times in the multiset, then the **multiplicity** of the element in the multiset is $k$.

# Multiset [多重集]

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## Examples:

In the multiset $\#\{a, a, b\}$, the element $a$ has multiplicity 2, and $b$ has multiplicity 1.

In the multiset $\#\{a, a, a, b, b, b\}$, $a$ and $b$ both have multiplicity 3.

The set $\{a, b\}$ contains only elements $a$ and $b$.
Each having multiplicity 1 when $\{a, b\}$ is seen as a multiset $\#\{a, a, a, b, b, b\}$.

**Order does not matter**: $\#\{a, a, b\}$ and $\#\{a, b, a\}$ denote the same multiset.

$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$ has eigenvalues $1, 1, 2$

$A$ has eigenvalues $1, 2$.

$B = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ has eigenvalue $1, 2$.

<u>Can we say</u>: $A$ and $B$ have the same eigenvalues?

<u>No</u>, if "have the same eigenvalues" means "have the same multiset of eigenvals".

# Linear Space over Complex Domain

## Recall:

Definition of Linear space over $\mathbb{R}$.

scalar product

$\alpha \cdot v, \ \alpha \in \mathbb{R}, \ v \in V$.

A linear space over $\mathbb{C}$ is defined in a similar way:
just change $\in \mathbb{R}$ to $\in \mathbb{C}$

scalar product

$\alpha \cdot v, \ \alpha \in \mathbb{C}, \ v \in V$

For completeness, we state the full definition in the next slide.

e.g. $i \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} i \\ 2i \end{pmatrix}$

$(1+i) \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} (1+i)\cdot 2 & \_ \\ \_ & \_ \end{bmatrix}$

# Reading: Def: Linear Space over Complex Domain

## Definition 23.3 (Linear space over $\mathbb{C}$)

Suppose V is a set associated with two operations:
(i) Addition "+": $\mathbf{u}+\mathbf{v} \in V, \forall \mathbf{u} \in V, \mathbf{v} \in V$.
(ii) Scalar multiplication: $\alpha\mathbf{u} \in V, \forall \alpha \in \mathbb{C}, \mathbf{u} \in V$.

V is called a linear space over $\mathbb{C}$ if the 8 axiom axioms hold:

(A1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \forall\ \mathbf{u},\ \mathbf{v} \in V$ .

(A2) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+\mathbf{v}+\mathbf{w}, \forall\ \mathbf{u},\ \mathbf{v},\ \mathbf{w}\ \in V$.

(A3) There exists a element $\mathbf{0}$ s.t. $\mathbf{u}+\mathbf{0}=\mathbf{u}, \forall\ \mathbf{u} \in V$.

(A4) If $\mathbf{u} \in V$, then there exists $-\mathbf{u}=(-1)\mathbf{u}$, s.t. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(A5) $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}, \forall\ \alpha \in \mathbb{C},\ \mathbf{u},\ \mathbf{v} \in V$.

(A6) $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}, \forall\ \alpha,\ \beta \in \mathbb{C},\ \mathbf{u} \in V$.

(A7) $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}, \forall\ \alpha,\ \beta \in \mathbb{C},\ \mathbf{u} \in V$.

(A8) $1\mathbf{u}=\mathbf{u}$.

e.g.,

$\mathbb{C}^n$ is linear space

$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad z_j \in \mathbb{C},$

## Space $\mathbb{C}$

Complex number written as $z = a + bi,$ where $i = \sqrt{-1}$

Complex conjugate of $x$ [$x$的共轭复数] is $\bar{z} =$ $a - bi$

Modulus of $x$ $z$ [$x$ $z$的模] is $|z| =$ $\sqrt{a^2+b^2}$ [using $a, b$]

$=$ $z \cdot \bar{z}$ [using $z, \bar{z}$]

A complex number $x = a + bi,$ where $i = \sqrt{-1}$

is a real number

iff $b=0$, iff $x = \bar{x}$.

## Space $\mathbb{C}^n$

**A few statements about the complex vector space.**

$\{e_1, e_2, e_3\}$ is still a basis of $\mathbb{C}^3$.

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = z_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} i \\ -1 \end{bmatrix}$ are linearly dependent.

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \cdot i = \begin{bmatrix} i \\ i^2 \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

If $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then $\sqrt{\bar{\mathbf{z}}^\top \mathbf{z}} = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$

$$\bar{z}^\top z = [\bar{z}_1, \cdots \bar{z}_n] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \bar{z}_1 z_1 + \cdots + \bar{z}_n z_n = |z_1|^2 + \cdots + |z_n|^2.$$

**Corollary:** $\mathbf{z} = \mathbf{0}$ iff $\bar{\mathbf{z}}^\top \mathbf{z} = 0$.

Judgement, $z \in \mathbb{C}^n$.

$z = 0 \Leftrightarrow z^\top z = 0$,

False.

# Big Picture About Eigenvalues & Eigenvectors

① exactly $n$ complex eigenvalues.

(luckily).

~~②~~ How many eigenvectors? $\infty$-many; not interesting.

③ How many independent eigenvectors?

at most $n$; but is it always $n$?

# How Many Linearly Independent Eigenvectors?

**Judgement**:

**Statement 1**: An $n \times n$ matrix always has $n$ linearly independent eigenvectors.

$\mathbb{R}^n$

$n$ vec

basis

indep

basis

span $\mathbb{R}^n$

basis.

## Example 1: Do Eigenvectors Form a Basis?

Example

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

Two eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$

(1). For $\lambda_1 = 4$,

For any $t \neq 0$, $x = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector with respect to $\lambda_1 = 4$

(2). For $\lambda_2 = 3$,

For any $t \neq 0$, $x = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector with respect to $\lambda_2 = 3$

**Do $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ form a basis of $\mathbb{R}^2$?**

Are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ independent?

95% yes

95% yes

# Example 2: Do Eigenvectors Form a Basis?

Example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Two eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$

(1). For $\lambda_1 = i$,

For any $\underline{t \neq 0}$, $x = t\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector with respect to $\underline{\lambda_1 = i}$

(2). For $\lambda_2 = -i$,

For any $\underline{t \neq 0}$, $x = t\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector with respect to $\underline{\lambda_2 = -i}$

**Do $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ form a basis of $\mathbb{C}^2$?** Yes

# How Many Linearly Independent Eigenvectors?

**Judgement**:

**Statement 1**: An $n \times n$ matrix always has $n$ linearly independent eigenvectors.

You may ask: real or complex?

**Judgement**:

**Statement 2**: An $n \times n$ real matrix always has $n$ linearly independent real eigenvectors.

False (by example 2, no real eigenvectors for some matrix)

**Judgement**:

**Statement 3**: An $n \times n$ real matrix always has $n$ linearly independent complex eigenvectors (note: including real eigenvectors).

False (by example 3, next page)

# Example 3: Do Eigenvectors Form a Basis?

Example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenvalues are $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 = 0$

$\Rightarrow \lambda = 1.$

Eigenvectors are $\begin{pmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow x_2 = 0$

$\mathrm{Null}(\lambda I - A) = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \middle| t \in \mathbb{C} \right\}.$

**Can you find two eigenvectors that form a basis of $\mathbb{R}^2$?** ($\mathbb{C}^2$)

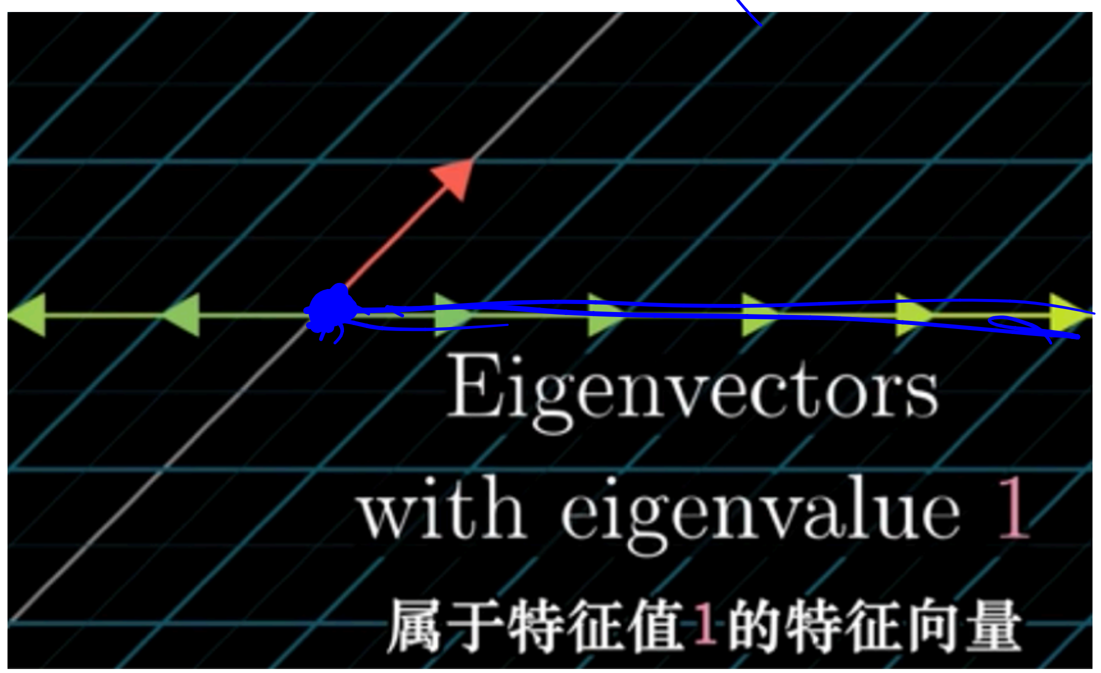
No. Cannot span $\mathbb{R}^2$ (or $\mathbb{C}^2$)

Remark (advanced reading) This example is related to "Jordan form"

Interested students may search it; we do NOT teach it.

# Eigenbasis may NOT exist

**Fact**: The (complex) eigenvectors of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ do NOT form a basis.




(1,0)

Eigenspace dimension dim(N($\lambda_1 I - A$)) =1.

Multiplicity is 2.

# Do Eigenvectors Form a Basis?

**Eigenbasis:**
**A set of e**igenvectors that can form a basis of whole space. ($\mathbb{R}^n$ or $\mathbb{C}^n$).

**Fact:** 某些
For certain $n \times n$ matrices, the eigenbasis does NOT exist.

即 i.e. can find at most $\underline{\ (n-1)\ }$ independent eigenvectors of $A$.

For some $n \times n$ matrices, eigenvectors can span $\mathbb{C}^n$
(can form a basis of $\mathbb{C}^n$).

# Part II Spectral Decomposition

Sec. 6.4

for real symmetric matrices.

**Lemma 23.1** If $Av_1 = \lambda_1 v_1, \cdots, Av_n = \lambda_n v_n$,

where $A \in \mathbb{C}^{n\times n}$, $v_j \in \mathbb{C}^{n\times 1}$, $\lambda_j \in \mathbb{C}$, $j=1,\cdots,n$,

then $AV = VD$, where $V = [v_1, \cdots, v_n] \in \mathbb{C}^{n\times n}$, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

**Proof:**

$$\begin{aligned} AV &= A[v_1, \cdots, v_n] \\ &= [Av_1, \cdots, Av_n] \\ &= [\lambda_1 v_1, \cdots, \lambda_n v_n] \qquad (**) \\ &= [v_1, \cdots, v_n]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \\ &= VD. \end{aligned}$$

$\square$.

**Lemma 23.2.** If $AV = VD$, where $V = [v_1, \cdots, v_n] \in \mathbb{C}^{n\times n}$, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$,

then $Av_j = \lambda_j v_j$, $j=1,\cdots,n$.

**Proof:** Reverse the chain of equalities in $(**)$.

# Orthonormal Eigenbasis

**Claim 23.1 [orthonormal eigenbasis ==> symmetric matrix]**

Eigenvectors of a real matrix $A$ can form an orthonormal basis of $\mathbb{R}^n$.

property of A

<==> $A = VDV^\top$ where $D$ is a real diagonal matrix, $V$ is a real orthogonal matrix.

Proof: "$\Rightarrow$" Suppose eigenvector $v_1, \cdots, v_n$ form an orthonormal basis of $\mathbb{R}^n$, then $V = [v_1, \cdots, v_n]$ is an orthonormal matrix.

Suppose $Av_j = \lambda_j v_j$, where $\lambda_j \in \mathbb{C}$; then $\lambda_j \in \mathbb{R}$ (since $A, v_j \in \mathbb{R}$)

By Lemma 23.1, we have $AV = VD$, where $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

Then $A = VDV^{-1} = VDV^\top$.

"$\Leftarrow$" Suppose $V = [v_1, \cdots, v_n]$, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Then $\{v_j\}$'s form an orthonormal basis.

By Lemma 23.2, we have $Av_j = \lambda_j v_j$, $j = 1, \cdots, n$. Thus $\{v_j\}$'s are eigenvectors!

$\Rightarrow$ eigenvectors of A can form an orthonormal basis. $\square$

**Corollary**: If a real matrix A has an orthonormal eigenbasis, then A is a symmetric matrix.

What about the other direction?

Proof sketch of Claim 23.1.

$$\boxed{\begin{aligned}&\underline{\text{Fact}} \quad A v_j = \lambda_j v_j, \ \forall j \in \{1, \dots, n\} \\ &\Rightarrow AV = VD.\end{aligned}}$$

<u>Next</u>, $\{v_j\}$ orthonormal $\Rightarrow V = [v_1, \dots, v_n]$

is <u>orthogonal</u> matrix.

$AV = VD$

$\Rightarrow \underline{A} = VDV^{-1} = \underline{VDV^T}$

<u>Proof of Corollary</u>:

$\underline{A^T} = (VDV^T)^T = (V^T)^T \underline{D^T} V^T$

$= VDV^T = A.$

$A = VV^T$ is symmetric

since $A^T = (V^T)^T V^T = VV^T = A$.

$A = VDV^T$ is also symmetric

(if $D$ is diagonal)

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} A : \text{ multiply } j\text{-th row by } \lambda_j$$

$$A \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} : \text{ multiply } j\text{-th column by } \lambda_j$$

# Spectral Theorem

**Theorem 23.1 [Spectral Theorem]**

Any real symmetric matrix $A$ can be written as

$$A = VDV^\top \quad (*)$$

where $D$ is a real diagonal matrix, $V$ is a real orthogonal matrix.

$Av_j = \lambda_j v_j$ (*)

$V = [v_1, \cdots, v_n]$, $v_j$'s are eigenvectors

$D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, $\lambda_j$'s are eigvals

**Vector form [practice]:**

$$A = VDV^\top = [v_1, \cdots, v_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$ [column & row form]

$$= \underbrace{[\lambda_1 v_1, \cdots, \lambda_n v_n]}_{1\times n} \underbrace{\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}}_{n\times 1}$$ [outer product]

(**)

$$= \lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T = \sum_{j=1}^{n} \lambda_j v_j v_j^T.$$

(*) and (**): Eigenvalue decomposition (EVD) or eigendecomposition of A.

# Remark on Vector Form of EVD

Matrix product (Lec 5-6).

$$AB = \sum_{j=1}^{n} a_j b_j^T \quad \text{(sum of outer products)}$$

Mid-term exam:

Express $BB^T = I$ by column form $B = [v_1, \cdots, v_n]$

$$I = BB^T = \underbrace{[v_1, \cdots, v_n]}_{n \times n} \begin{bmatrix} \underbrace{v_1^T}_{n \times 1} \\ \vdots \\ v_n^T \end{bmatrix} = \sum_{j=1}^{n} v_j v_j^T,$$

Here: $A = VDV^T = \sum_{j=1}^{n} \lambda_j v_j v_j^T$, (weighted sum of outer products), Two ways to check $v_j$'s are eigenvec's.

<u>Method 1.</u> By Lemma 23.2

<u>Method 2:</u>

If $A = \sum_{j=1}^{n} \lambda_j v_j v_j^T$, and $\{v_j's\}$ orthonormal set,
($v_j \perp v_k$, $\forall j \neq k$,
$\|v_j\| = 1$).

then $A v_j = \lambda_j v_j$.

<u>Proof.</u> $A \cdot v_1 = (\lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T) \cdot v_1$

$= \lambda_1 v_1 \underline{v_1^T v_1} + \lambda_2 v_2 v_2^T v_1 + \cdots + \lambda_n v_n \underline{v_n^T v_1}$

$= \lambda_1 v_1 + 0 + \cdots + 0$

$= \lambda_1 v_1$.

# Properties of Real Symmetric Matrices

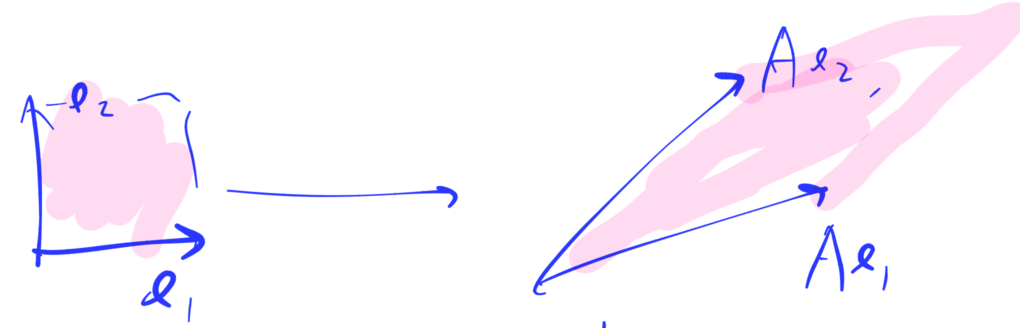
| Real Symmetric Matrices | General Matrices |
|---|---|
| **Property 1**: All eigenvalues are real. | Eigenvalues can be complex. |
| **Property 2**: All eigenvectors are real. (more rigorously, can be real) | Eigenvectors can be complex. |
| **Property 3**: Eigenvectors can form an orthonormal basis of $\mathbb{R}^n$. (or $\mathbb{C}^n$) | Eigenvectors may NOT form basis of $\mathbb{C}^n$. |
| **Together**: $A$ can be written as | A can be written as ?? |

$$A = VDV^T = \sum_{j=1}^{n} \lambda_j v_j v_j^T.$$

# Geometry View

Previous lectures

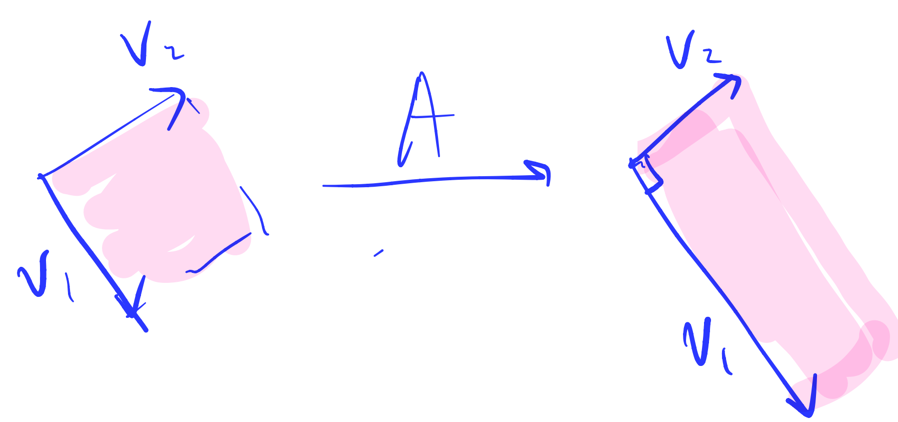
Basis $\{e_1, e_2\}$




shape of tiles change

Here: For symmetric $A \in \mathbb{R}^{n \times n}$:

Basis $\{v_1, v_2\}$




$x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} (x_1, x_2)$

shape does not change (too much)

directions stay.

$Ax = (v_1, v_2)(\lambda_1 x_1, \lambda_2 x_2).$

## Proof of Spectral Theorem

The full proof of spectral theorem is skipped here
(It requires a bit trick and induction)

$\lambda_j \neq \lambda_k, \ \forall j \neq k,$

Next, we only prove the case for distinct eigenvalues.

only prove the case for distinct eigenvalues.

only prove the case for distinct eigenvalues.

## Proof of EVD for Distinct Eigenvalues (I): Real Eigenvalues

**Assumption 23.1**: $A \in \mathbb{R}^{n\times n}$ and $A = A^\top$.

**Property 23.1**: Under Assump. 23.1, all eigenvalues are real.

### Analysis:

What we know:

Suppose $A\mathbf{v} = \lambda \mathbf{v}$, ← play with it by multiplying sth.

where $\lambda \in \mathbb{C}, \mathbf{v} \in \mathbb{C}^{n\times 1}\backslash\{\mathbf{0}\}$.

Want to prove: $\lambda \in \mathbb{R}$.

$A = A^T \Rightarrow v^T A = v^T A^T = \lambda v^T$ ①

$A = \bar{A} \Rightarrow A\bar{v} = \bar{A}\bar{v} = \bar{\lambda}\bar{v}$ ②

Combine $v^T\boxed{=}$ and $\bar{A}\bar{v}$

# Proof of EVD for Distinct Eigenvalues (I): Real Eigenvalues

**Assumption 23.1**: $A \in \mathbb{R}^{n\times n}$ and $A = A^\top$. Fact: $\bar{v}^T v = 0$ iff $v = 0$.

$v \neq 0 \Rightarrow \bar{v}^T v \neq 0$ ③

## Proof of Property 23.1:

Suppose $A\mathbf{v} = \lambda\mathbf{v}$, where $\lambda \in \mathbb{C}, \mathbf{v} \in \mathbb{C}^{n\times 1}$. Want to prove: $\lambda \in \mathbb{R}$.

Use two ways to compute $\bar{\mathbf{v}}^\top A\mathbf{v}$. 算二次

First way: $\bar{\mathbf{v}}^\top A\mathbf{v} = \bar{v}^T (A\cdot v) = \bar{v}^T \cdot \lambda \cdot v = \lambda \bar{v}^T v$

Second way: $\bar{\mathbf{v}}^\top A\mathbf{v} = (\bar{v}^T A) v = (\bar{v}^T A^T) v$

symmetric

$= (A\bar{v})^T v$

real

$= (\bar{A}\bar{v})^T v$

$\overset{②}{=} \bar{\lambda} \cdot \bar{v}^T v.$

③ $\Rightarrow$ $\lambda = \bar{\lambda}$.

i.e. $\lambda \in \mathbb{R}$.

# Proof of EVD for Distinct Eigenvalues (I): Real Eigenvectors

**Assumption 23.1 (1st part)**: $A \in \mathbb{R}^{n \times n}$.

**Property 23.2**: For any real eigenvalue of A, there exists a real eigenvector with respect to this eigenvalue.

## Proof of Property 23.2:

Suppose $A\mathbf{v} = \lambda\mathbf{v}$, where $\lambda \in \mathbb{R}$, $\mathbf{v} \in \mathbb{C}^{n\times 1} \backslash \{\mathbf{0}\}$.

Suppose $v = x + y \cdot i$,

where $x, y \in \mathbb{R}^{n\times 1}$; one of $x, y$ is $\neq 0$, ①

$A(x+yi) = \lambda(x+yi)$

$\Leftrightarrow \underbrace{A}_{\text{real}}\underbrace{x}_{\text{real}} + \underbrace{A}_{\text{real}}\underbrace{y}_{\text{real}} \cdot i = \underbrace{\lambda x + \lambda y}_{\text{Prop 23.1}} \cdot i$

$\Leftrightarrow Ax = \lambda x,\ Ay = \lambda y$. ②

① & ②

$\Rightarrow$ One of $x, y$ should be eigenvectors

# Proof of EVD for Distinct Eigenvalues (III):Orthogonality

**Assumption 23.1**: $A \in \mathbb{R}^{n \times n}$ and $A = A^\top$.

**Property 23.3**: Under Assumption 23.1, real eigenvectors corresponding to different eigenvalues are orthogonal.

# Proof of EVD for Distinct Eigenvalues (III):Orthogonality

**Assumption 23.1**: $A \in \mathbb{R}^{n\times n}$ and $A = A^\top$.

Property 23.3 Under Assumption 23.1, eigenvectors corresponding to different eigenvalues are orthogonal. want $\Rightarrow x_1^T x_2 = 0$

**Proof**: Suppose $\underbrace{Ax_1 = \lambda_1 x_1}_{①}$, $\underbrace{Ax_2 = \lambda_2 x_2}_{②}$, $\lambda_1 \neq \lambda_2$. Check $x_1^\top A x_2$. 算两次!

$$\underbrace{x_1^T}_{1\times n}\ \underbrace{A}_{n\times n}\ \underbrace{x_2}_{n\times 1}$$

$$x_1^T A x_2 = (x_1^T A)\, x_2 \overset{\text{symmetric}}{=} (x_1^T A^T)\, x_2 \overset{①}{=} \lambda_1 x^T x$$

$$x_1^T A x_2 = x_1^T (A x_2) \overset{②}{=} \lambda_2 x_1^T x_2$$

$$\Rightarrow x_1^T x_2 = 0,$$

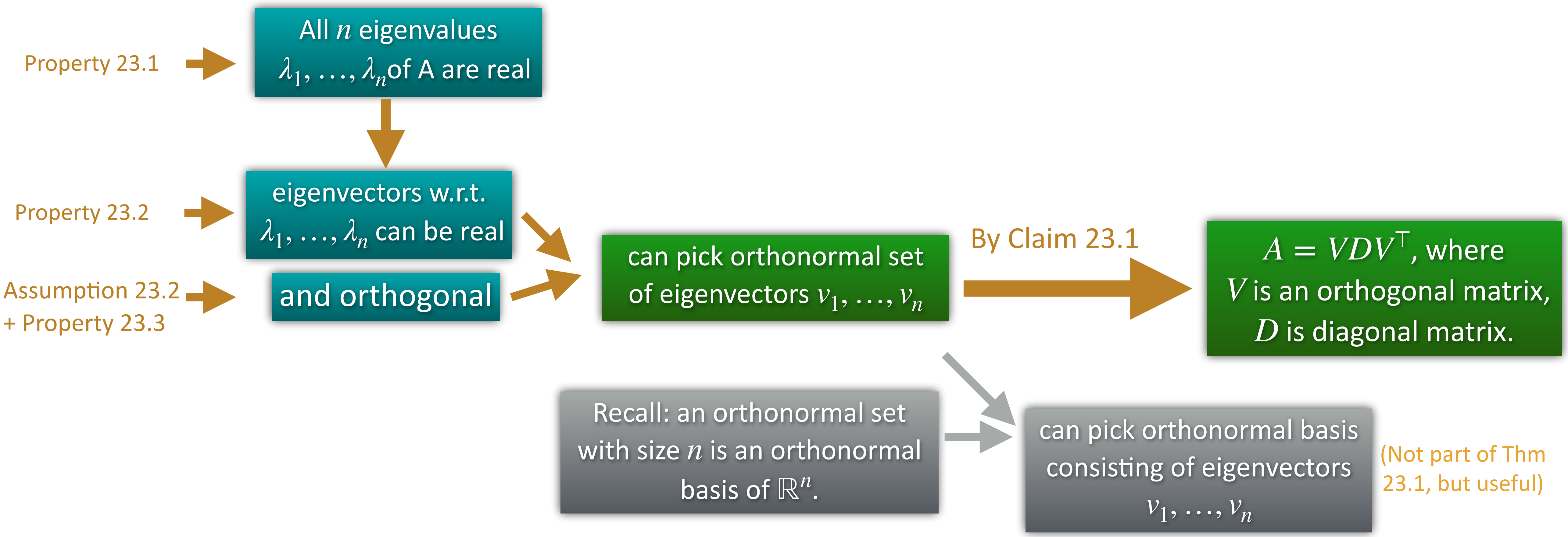
1 2

# EVD for Distinct Eigenvalues

**Assumption 23.1**: $A \in \mathbb{R}^{n\times n}$ and $A = A^\top$.

**Assumption 23.2**: all eigenvalues of $A \in \mathbb{R}^{n\times n}$ are distinct.

**Proof of Thm 23.1 under Assumption 23.2: (Combining Property 23.1,23.2, 23.3)**




So far, we have proved Thm 23.1 for distinct eigenvalues.

The proof for general case "any real symmetric matrix A can be written as $A = VDV^\top$" is SKIPPED.

# Summary Today (Write Your Own)

**One sentence summary:**

**Detailed summary:**

# Summary Today (Instructor)

**One sentence summary:**

We learned eigenbasis and spectral decomposition.

**Detailed summary:**

i) For some matrices, there are $n$ eigenvectors that form a basis, called eigenbasis.

**Fact**: Eigenbasis may or may NOT exist, for a given matrix.

ii) Eigenvalue decomposition (特征值分解), or spectral decomposition.

A real symmetric matrix $A$ can be written as

$$A = VDV^\top = \sum_{j=1}^{n} \lambda_j v_j v_j^\top$$

where