Lecture 24

Eigenvalue III: Properties

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Today's Lecture: Outline and Learning Goals

Main topic: Properties

1. Properties: Sum, Product of Eigenvalues
2. Similar matrices

After the lecture, you should be able to

1. Prove and apply a few properties of eigenvalues
2. Tell the property of similar matrices

Epidemic [流行病]

Review

Judgement Question

build questions yourself

Q1: Any real square matrix A can be written as $A = SDS^\top$ where D is a diagonal matrix, and S is an orthogonal matrix. False

SDS^T is symmetric; if A is not symmetric, then $A \neq SDS^T$.

Q2: If there are n eigenvectors of a real matrix A that can form an orthonormal basis of $\mathbb{R}^n$, then A is a symmetric matrix. LD3: 40% true.

True. (from previous lecture)

Q3: For any real square matrix A, we can find a diagonal matrix D (possibly complex) and a square matrix V (possibly complex), such that $AV = VD$. True.

few people voted.

Eigenvalues and Eigenvectors

Definition 21.1 (Eigenvalues and Eigenvectors)

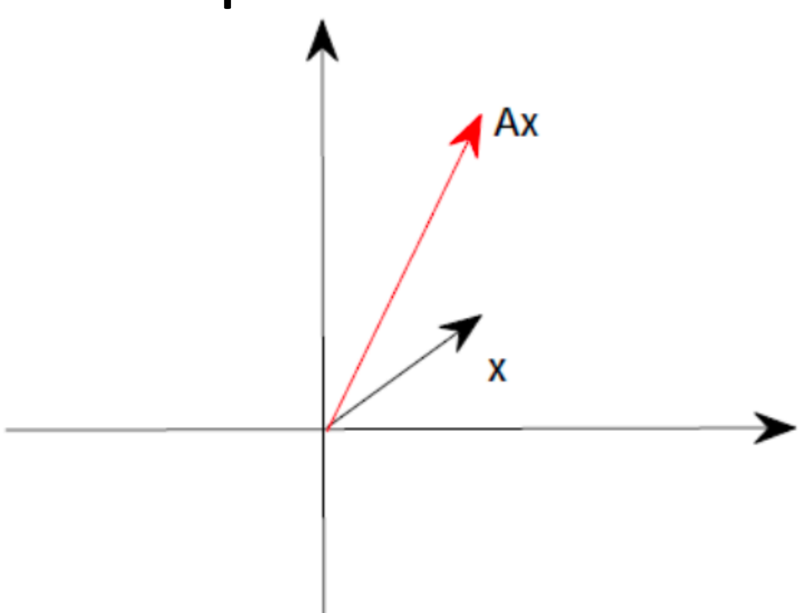
Let $A \in \mathbb{C}^{n \times n}$ be a square matrix.

If there exists a scalar λ ($\in \mathbb{R}$ or $\mathbb{C}$) and a **nonzero** vector x such that

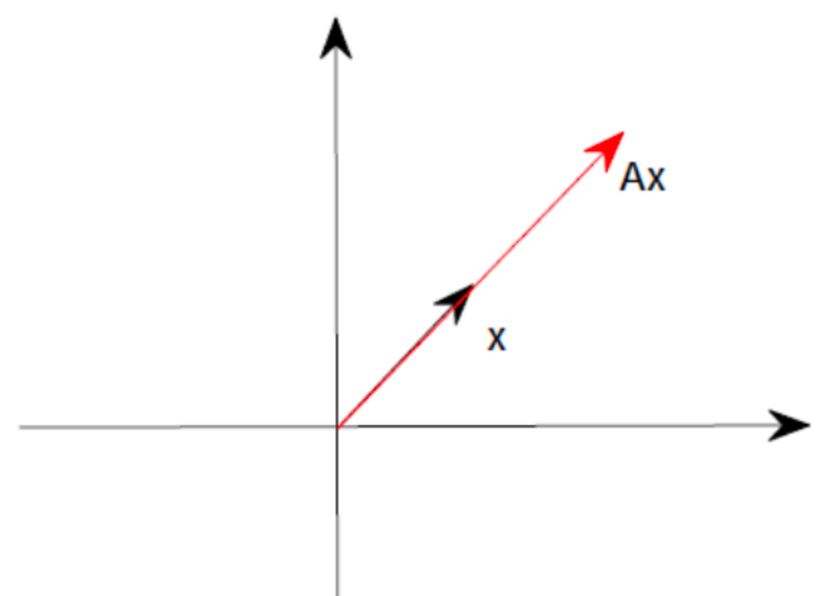
$$Ax = \lambda x,$$

then λ is called a (real or complex) **eigenvalue** and x is called an **eigenvector** with respect to (or associated with; corresponding to) λ.

$\mathbb{C}$:= set of complex numbers



x is **not** an eigenvector



x is an eigenvector

General Procedure to Find Eigenvalues/Eigenvectors

Step 1: Solve $\det(A - \lambda I) = 0$ and get n roots $\lambda_1, \ldots, \lambda_n$

Solving single-var polynomial.

Step 2: For each λ_i, find the eigenspace $\mathrm{Null}(\lambda_i I - A)$

Solving up to n linear systems.

Any nonzero vector in $\mathrm{Null}(\lambda_i I - A)$ is an eigenvector corresponding to λ_i

$$\mathrm{Null}(\lambda_i I - A) = \left\{ \begin{array}{l} \text{eigenvectors of } A \text{ with} \\ \text{respect to the eigenvalue } \lambda_i \end{array} \right\} \bigcup \; \{0\}$$

Fact:

Any $n \times n$ (no matter real or complex) matrix A has n **complex eigenvalues** (counting multiplicity).

Characteristic polynomial $p_\lambda(A) = \det(A - \lambda I)$ has n roots.

Spectral Theorem

Theorem 23.1 [Spectral Theorem]

Any real symmetric matrix A can be written as

$$A = VDV^\top = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^\top . \qquad (*)$$

where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is a real diagonal matrix,

$V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is a real orthogonal matrix.

(*): Eigenvalue decomposition (EVD) or eigendecomposition of A.

Property 23.1: All eigenvalues are real.

Property 23.2: Eigenvectors can form an orthonormal basis of $\mathbb{R}^n$.

(23.2a) Eigenvectors can form a basis of $\mathbb{R}^n$.

(23.2b) Can pick an eigenbasis that is orthonormal set.

Part I Sum and Product of Eigenvalues

Play with $p_A(\lambda)$

matrix

polynomial

Eigenvalues are roots of characteristic polynomial.

We will explore results on polynomials,
to get some properties of eigenvalues.

Review of Middle School Results: Vieta's Theorem

Vieta's Theorem (韦达定理):

If $ax^2+bx+c=0$ has two roots x_1, x_2

then $x_1+x_2=-\frac{b}{a}$, $x_1x_2=\frac{c}{a}$.

Can you prove it?

One way is to use root formula.

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \quad x_1+x_2=-\frac{b}{a}, \quad x_1x_2=\frac{c}{a}$$

What about 3rd order equation?

$$ax^3+bx^2+cx+d=0$$

$$x_1+x_2+x_3=? \quad x_1x_2x_3=?$$

For order ≥ 5 equations, root formulas do NOT exist

Better Proof

Claim If $ax^2+bx+c=0$ has two roots x_1, x_2, $(a\neq 0)$

then $ax^2+bx+c = a(x-x_1)(x-x_2)$. ①

[This claim can be proved by showing $x-x_i$ is a factor of ax^2+bx+c]

Proof Method 2 of Vieta's theorem for quadratic equation:

Express RHS as $a\left(x^2-(x_1+x_2)x+x_1x_2\right)$

$$= ax^2 - \underline{a(x_1+x_2)}\,x + \underline{a\,x_1x_2} \quad ②$$

Compare ________, get

coefficients of LHS of ① & RHS of ②

coefficient of x: $b = -a(x_1+x_2) \Rightarrow x_1+x_2 = \frac{-b}{a}$.

← ——— of 1 (x^0): $c = ax_1x_2 \Rightarrow x_1x_2 = \frac{c}{a}$

Vieta's Theorem: General Case

Vieta's Theorem (deg-n case)

Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has n roots $(a_n \neq 0)$ $x_1, x_2, \cdots, x_n$, then $\sum_{j=1}^{n} x_j = -\frac{a_{n-1}}{a_n}$, ①

$$\prod_{j=1}^{n} x_j = \frac{a_0}{a_n}(-1)^n. \quad ②$$

$$\prod_{j=1}^{m} (x - x_j)^{k_j}$$

Proof of Vieta's theorem:

Express f(x) as $f(x) = a_n (x - x_1)(x - x_2) \cdots (x - x_n)$.

$$= a_n \left(x^n - (x_1 + x_2 + \cdots + x_n) x^{n-1} + \cdots + (-1)^n x_1 \cdots x_n \right)$$

Compare coefficient, get

coefficient of x^{n-1}: $a_{n-1} = -a_n (x_1 + \cdots + x_n)$

$$\Rightarrow \sum_{j=1}^{n} x_j = -\frac{a_{n-1}}{a_n}$$

②

coeff— of x^0 (1), $a_0 = a_n \cdot (-1)^n x_1 \cdots x_n \Rightarrow \prod_{j=1}^{n} x_j = \frac{a_0}{a_n}(-1)^n$

Reading: Proof of Decomposition of $f(x)$

<u>Lemma.</u> If x_k is a root of $f(x) = \sum_{j=0}^{n} a_j x^j$,
the $f(x) = (x - x_k) \cdot h(x)$, where degree of $h(x) \leq n-1$.

<u>Proof.</u> Prove by induction. Assume the result holds for $n-1$.
Consider degree n.

$$f(x) = (x - x_k) \cdot h_0(x) + \underbrace{f_1(x)}_{\deg \leq n-1}$$

$$\Big\downarrow x = x_k$$

$$0 = f(x_k) = f_1(x_k).$$

By induction hypothesis, $f_1(x_k) = h_1(x) \cdot (x - x_k)$,
Thus $f(x) = (x - x_k)\big(h_0(x) + h_1(x)\big)$. Thus the result holds for n.

Applying Vieta's Theorem to $p_A(\lambda)$: 1st Calculation of Coefficients

depend on A.

$$p_\lambda(A) = \det(A - \lambda I) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-1}\lambda^{n-1} + (-1)^n\lambda^n.$$

Roots are eigenvalues of A: $\lambda_1, \ldots, \lambda_n$.

sum of roots of $p_\lambda(A)$

By Vieta's theorem, the sum of eigenvalues is $\underline{\quad -\frac{\alpha_{n-1}}{\alpha_n} \quad}$ $= (-1)^{n-1}\alpha_{n-1}$

$(-1)^2$ $/\!/$

By Vieta's theorem, the product of eigenvalues is $\underline{\quad (-1)^n\frac{\alpha_0}{\alpha_n} \quad}$ $= \alpha_0$.

What are the coefficients of $p_\lambda(A)$?

2nd Calculation of Coefficients of $p_A(\lambda)$

λ^{n-1}

Consider n=3 case: Write $A = (a_{ij})_{3\times 3}$

coefficients of λ^2, $\lambda^0 (=1)$

Use definition of determinant, to expand:

$$p_\lambda(A) = \det(A - \lambda I) = \det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$= (a_{11}-\lambda)\det(C_{11}) - a_{12}\det(C_{12}) + a_{13}\det(C_{13})$$

each term is a product of 3 entries which are not in the same row & column

$$= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) - (a_{11}-\lambda)a_{23}a_{32} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda \end{vmatrix}}_{\text{no } \lambda^2} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{no } \lambda^2}$$

There're many terms that contain $\lambda^0 = 1$ (constant terms), so we need a different method to compute α_0 [next page] (sum of eigenvalues) $/(-1)^{n-1}$

$= $ coefficient of λ^2 in $p_A(\lambda)$

$= $ coefficient of λ^2 in

$$\left(\begin{aligned} &(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) \\ &= (-1)^3\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) + \cdots \end{aligned} \right)$$

$= a_{11} + a_{22} + a_{33}$

$=$ sum of diagonal entries.

Product of Eigenvalues

$f(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^{n-1}$

$f(0) = a_0$.

$$p_\lambda(A) = \det(A - \lambda I)$$

Roots are eigenvalues of A: $\lambda_1, \ldots, \lambda_n$.

$n = 3$ case:

$\det(A - \lambda I_3) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \cdot (-1)^3$

pick $\lambda = 0$

pick $\lambda = 0$

$\det(A) = \lambda_1 \lambda_2 \lambda_3$

n=3 case and General Expressions

$$p_\lambda(A) = \det(A - \lambda I)$$

Roots are eigenvalues of A: $\lambda_1, \ldots, \lambda_n$.

$n = 3$ case:

$$\lambda_1 + \lambda_2 + \lambda_3 = \sum_{j=1}^{3} a_{jj}$$

$$\lambda_1 \lambda_2 \lambda_3 = \det(A)$$

General n case:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{j=1}^{n} a_{jj}$$

$$\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$$

Summary of Derivation

How did we prove $\sum_{j=1}^{3} \lambda_j = \sum_{j=1}^{3} a_{jj}$ and $\prod_{j=1}^{3} \lambda_j = \det(A)$?

Knowledge

1) roots of $p_\lambda(A)$

2) Vieta Thm

3) Det's def

Logic

① Fact: λ_j's are roots of polynomial $p_\lambda(A)$

↓

② $\sum \lambda_j = \sum$ roots $\xrightarrow{\text{Vieta thm}}$ ratio of coefficients of $p_\lambda(A)$

↓

③ Compute coefficients in $p_\lambda(A) = \det(A - \lambda I)$ by definition of determinant.

Trick observe coeff rely on 1 term (coef of λ^2)
or 6 terms (coeft of 1)

This trick can save some time.

Knowledge.

Remark. I hope the summary can be done by yourself.

Sum of Eigenvalues

Question:

What is the sum of eigenvalues?

Proposition 24.1 (Sum of Eigenvalues)

Let $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) be a square matrix.

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

Sum of Eigenvalues

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$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

Definition: $\sum_{i=1}^{n} a_{ii}$ is called the **trace** of A, denoted by $\mathrm{tr}(A)$.

迹

Proof

1. **Definition of Eigenvalues:**
   Eigenvalues of a matrix A are the solutions to the characteristic equation $\det(A - \lambda I) = 0$, where λ represents an eigenvalue and I is the identity matrix of the same size as A.
2. **Characteristic Polynomial:**
   The characteristic polynomial of A is given by $p(\lambda) = \det(A - \lambda I)$. For an $n \times n$ matrix, this polynomial is of degree n and can be written as $p(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$, where the coefficients c_i are functions of the entries of A.

Remark:
Proof trick
"compute twice"
算两次

3. **Coefficient of λ^{n-1} in the Characteristic Polynomial:**
   By expanding $\det(A - \lambda I)$, it can be shown that the coefficient c_{n-1} of λ^{n-1} is (up to a sign) the sum of the diagonal entries of A, which is the trace of A.

$$C_{n-1} = tr(A) \cdot (-1)^{n-1} \quad \cdots \text{①}$$

4. **Sum of Roots of the Characteristic Polynomial:**
   According to Vieta's formulas, the sum of the roots of a polynomial (which are the eigenvalues in this case) is equal to the negation of the coefficient of the second-highest degree term of the polynomial divided by the leading coefficient. For the characteristic polynomial, this is $-c_{n-1}/(-1)^n$, which simplifies to the trace of A (since the leading coefficient is $(-1)^n$ for the λ^n term).

$$\sum_{j=1}^{n} \lambda_j = \frac{-C_{n-1}}{(-1)^n} = (-1)^{n-1} C_{n-1} \overset{①}{=} tr(A)$$

Therefore, the sum of the eigenvalues of matrix A is equal to the trace of A. This proof is quite abstract and involves a good understanding of matrix algebra, determinants, and polynomial theory.

Product of Eigenvalues

Question:

What is the product of eigenvalues?

Proposition 24. (Product of Eigenvalues)

Let $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) be a square matrix.

$$\prod_{i=1}^{n} \lambda_i = \det(A)$$

Simple Matrices Checking: Diagonal Matrices

$$A = \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{pmatrix}$$

Eigenvalues of A are $q_1, \dots, q_n$ ($\lambda_1 = q_1, \dots, \lambda_n = q_n$).

Check.

$$\mathrm{tr}(A) = \sum_{j=1}^{n} q_j = \sum_{j=1}^{n} \lambda_j .$$

$$\det(A) = \prod_{j=1}^{n} q_j = \prod_{j=1}^{n} \lambda_j .$$

Check: Triangular matrix.

Example of Sum and Product of Eigenvalues

Example

$$A = \begin{bmatrix} 5 & -18 \\ 1 & -1 \end{bmatrix}$$

$\mathrm{tr}(A) = 5 + (-1) = 4$

$\det(A) = 5 \cdot (-1) - 1 \cdot (-18) = 13.$

Next, we'll check whether

$\lambda_1 + \lambda_2 = 4,$

$\lambda_1 \lambda_2 = 13.$

Example of Sum and Product of Eigenvalues

Example

$$A = \begin{bmatrix} 5 & -18 \\ 1 & -1 \end{bmatrix}$$

$\det(A) = 13$ and $\mathrm{Trace}(A) = 4$.

$$p_A(\lambda) = |A - \lambda I| = \begin{bmatrix} 5-\lambda & -18 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 13$$

$\lambda_1 + \lambda_2 = 4$ $\lambda_1 \lambda_2 = 13$

$\lambda_1 = 2 - 3i, \lambda_2 = 2 + 3i.$

$\det(A) = \lambda_1\lambda_2 = \underbrace{4+9}_{2^2+3^2} = 13,\ Trace(A) = \lambda_1 + \lambda_2 = 4.$

Part II Similar Matrices

$A \xrightarrow{\text{tranform}} B$.

Q1 What transformation makes eigenvalues unchanged?

Similar high-level idea existed before.

① $v \xrightarrow{A} Av$, direction does not change

What question is similar to Q1?

Q2. What operation makes solutions (of linear system) unchanged?

$A \xrightarrow{\text{GJ-E}} R$, or $A \xrightarrow{\text{elementary row operation}} B$

Similar Matrices

Definition 24.1 (Similar matrix 相似矩阵)

If two $n \times n$ matrices A and B satisfy $A = S^{-1}BS$ for some $S \in \mathbb{R}^{n\times n}$,
then we say A and B are similar (相似).

$$A = S^{-1}BS$$

$$\Downarrow \quad \not\Uparrow \text{ (since } S^{-1} \text{ may or may not exist).}$$

$$BS = SA$$

For any square matrix A,

$\exists$ square matrix S, diagonal matrix D, s.t.

$$AS = SD.$$

(if S^{-1} $\exists$) $\Longrightarrow$ $A = S \cdot D S^{-1}$

$\Longrightarrow$ A is similar to diagonal matrix D.

Remark. For some A, you can find invertible S;

for some A you cannot find invertible S.

Eigenvalues of Similar Matrices

Proposition 24.3 (First version)

Suppose $A, B \in \mathbb{R}^{n\times n}$ and are similar, i.e., there exists an invertible matrix S such that $A = SBS^{-1}$, then A and B have the same eigenvalues.

What does "have the same eigenvalues" mean?

Does $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ "have the same eigenvalues"?

We want the answer to be "NO".

Multiset.

$\#\{1, 2, 2\}, \quad \#\{1, 1, 2\}.$

Product of Eigenvalues

not widely used

Notation: Denote EIG(A) as the multiset of eigenvalues of matrix A.

Proposition 24.3 (Precise version. Eigvals of Similar matrices)

Suppose $A, B \in \mathbb{R}^{n \times n}$ and are similar, i.e., there exists an invertible matrix S such that $A = SBS^{-1}$, then A and B have the same eigenvalues, i.e., EIG(A) = EIG(B).

Does $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ "have the same eigenvalues"?

No, since EIG(A) $\neq$ EIG(B).

Product of Eigenvalues

Proposition 24.3 (Similar matrices)

Suppose $A, B \in \mathbb{R}^{n\times n}$ and are similar, then EIG(A) = EIG(B).

(but useful & interesting)

Trek.

1) Associative rule;

2) Change of variable $x \to u$.

Proof (incorrect): By definition.

If λ is an eigenvalue of A, then for some $x \neq 0$,

$$Ax = \lambda x \Rightarrow SBS^{-1}x = \lambda x$$

$$\Rightarrow B\underbrace{(S^{-1}x)}_{=u} = \lambda \underbrace{(S^{-1}x)}_{=u} \Rightarrow Bu = \lambda u,$$

where $u = S^{-1}x \in \mathbb{C}^n$ & $u = S^{-1}x \neq 0$. $\Rightarrow \lambda$ is an eigenvalue of B

Thus, we have shown:

If λ is an eigenvalue of A, then λ is an eigenvalue of B ①

Similarly, we can prove: If λ is an eigenvalue of B, then λ is an eigenvalue of A. ②

① & ② $\cancel{\Rightarrow}$ EIG(A) = EIG(B)

Judgement Question: ① & ② $\Rightarrow$ $EIG(A) = EIG(B)$.

If λ is an eigenvalue of A, then λ is an eigenvalue of B ①

If λ is an eigenvalue of B, then λ is an eigenvalue of A. ②

Counter-example.

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & 0 \\ & 2 & \\ & & 2 \end{bmatrix}$$

① & ② hold.

$EIG(A) = \#\{1, 1, 2\} \neq EIG(B) = \#\{1, 2, 2\}$,

thus ③ does NOT hold.

Therefore, ① & ② are NOT enough to prove ③.

"New" result: (corollary of old results)

$\left.\begin{array}{l} x \text{ is nonzero} \\ S^{-1} \text{ is invertible.} \end{array}\right\} \Rightarrow S^{-1}x \neq 0.$

① $S^{-1} \cdot x = 0$ has unique solution $x = 0$.

② $S^{-1}x = \text{LC of col of } S^{-1} \neq 0.$

$P_\lambda(A) = (\lambda-\lambda_1)\cdots(\lambda-\lambda_n)(-1)^n$

$\#(\lambda_1, \cdots \lambda_n) \xleftrightarrow{\text{determined}} P_\lambda(A)$

Product of Eigenvalues

Proposition 24.3 (Similar matrices)

Suppose $A, B \in \mathbb{R}^{n\times n}$ and are similar, then EIG(A) = EIG(B).

Proof (correct): By the fact that roots of $P_\lambda(A)$ form EIG(A),

$EIG(A) = EIG(B)$

$\Leftrightarrow \quad P_\lambda(A) = P_\lambda(B)$

$\Leftrightarrow \det(A-\lambda I) = \det(B-\lambda I)$

$\overset{\text{similarity}}{=} \det(SAS^{-1}-\lambda I)$

typical proof

$\det(SAS^{-1}-\lambda I) = \det(S(A-\lambda I)S^{-1})$

$= \det(S)\det(A-\lambda I)\det(S^{-1})$

$= \det(A-\lambda I) \quad \square$

Tools

Think:
What's in your toolbox for eigenvalues?

1) Definition: $Ax=\lambda x$

2) Characteristic polynomial.

Exercise: Judgement

Judgement: If $A, B \in \mathbb{R}^{n\times n}$ are similar, then A and B have the same eigenvalues.

Judgement: If $A, B \in \mathbb{R}^{n\times n}$ have the same eigenvalues, then they are similar.

Judgement: A real square matrix is similar to a real diagonal matrix.

Judgement: If A and B are similar, then $tr(A) = tr(B)$.

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Summary Today (Instructor)

One sentence summary:

We learned similar matrix and properties of eigenvalues.

Detailed summary:

i) Sum of eigenvalues = trace of matrix.
ii) Product of eigenvalues = determinant of matrix.

iii) Similar matrix: $A = SBS^{-1}$

Property: Similar matrices have the same multiset of eigenvalues.

Appendix Diagonalizable

Recall: Collection of Eigen-equations

Collection of n equations for eigenvectors, can be written in matrix form as follows.

Lemma 23.1 [eigen-equations in matrix form]

Suppose $Av_1 = \lambda_1 v_1, \ldots, Av_n = \lambda_n v_n,$ where $v_j \in \mathbb{C}^n, \lambda_j \in \mathbb{C}$. Then $AV = VD,$ (*)

where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, $V = [v_1, \ldots, v_n]$.

Lemma 23.2 [reverse: matrix form to eigen-equations]

Suppose $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, $V = [v_1, \ldots, v_n]$, where $v_j \in \mathbb{C}^n, \lambda_j \in \mathbb{C}$.

Suppose $AV = VD.$ (*) Then $Av_1 = \lambda_1 v_1, \ldots, Av_n = \lambda_n v_n,$

Diagonalizable

Claim 24.1 [indep. e.v. ==> A has a special decomposition]

If an $n \times n$ matrix A has n linearly independent eigenvectors $v_1, \ldots, v_n$, corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$A = S^{-1}DS, \forall k,$$

where $S = [v_1, \ldots, v_n] \in \mathbb{R}^{n \times n}$ is invertible, $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

Claim 24.2 [indep. e.v. <== A has a special decomposition]

If an $n \times n$ matrix A can be written as $A = S^{-1}DS, \forall k$, where D is a diagonal matrix, then it has n linearly independent eigenvectors.

Similar and Diagonalizable 相似矩阵和可对角化

Definition 24.1 (Similar matrix 相似矩阵)

If two $n \times n$ matrices A and B satisfy $A = S^{-1}BS$ for some $S \in \mathbb{R}^{n \times n}$,
then we say A and B are similar (相似).

Definition 24.2 (Diagonalizable 可对角化)

If a square matrix A is similar to a diagonalizable, then we say A is diagonalizable.

Theorem 24.1 [Suffi. & Necc. Conditions for Diagonalizable: n indep. eigenvectors]
An $n \times n$ matrix A is diagonalizable iff eigenvectors of A can form a basis.

We call this basis an "eigenbasis" (corresponding to A).

Proof of 2*2 Case

Lemma: If $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A, and V = [$\mathbf{v}_1, \mathbf{v}_2$], then

$$AV = SV.$$

Thm 22.1 (n=2 case)

Eigenvectors of A can form a basis $\Longleftrightarrow A = S\text{diag}(\lambda_1, \lambda_2)S^{-1}$.

Proof of "$\Longrightarrow$": Suppose eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ form a basis, then

$S = [\mathbf{v}_1, \mathbf{v}_2]$ is invertible.

By Lemma, $AS = SD \quad \Rightarrow A = SDS^{-1}$.

Proof of "$\Longleftarrow$": Suppose $S = [\mathbf{v}_1, \mathbf{v}_2]$; since it's invertible, $\mathbf{v}_1, \mathbf{v}_2$ form a basis.

$A = S\text{diag}(\lambda_1, \lambda_2)S^{-1} \Rightarrow AS = S\text{diag}(\lambda_1, \lambda_2) \Rightarrow A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

Appendix Applications

Application : PageRank

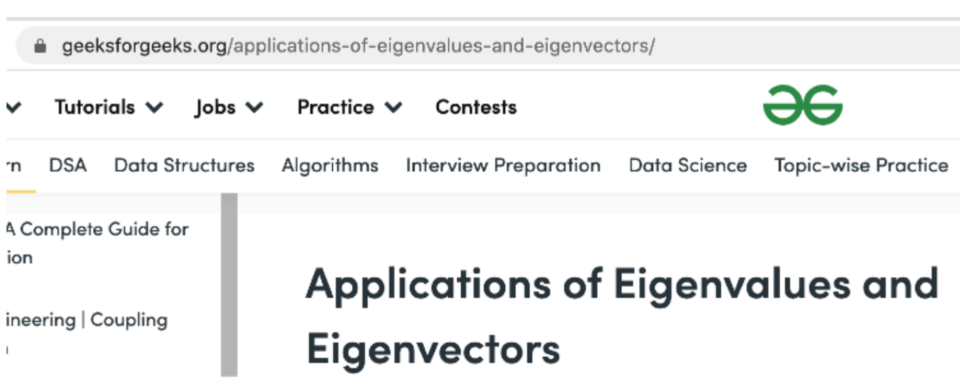
(Google’s 1st Algorithm)

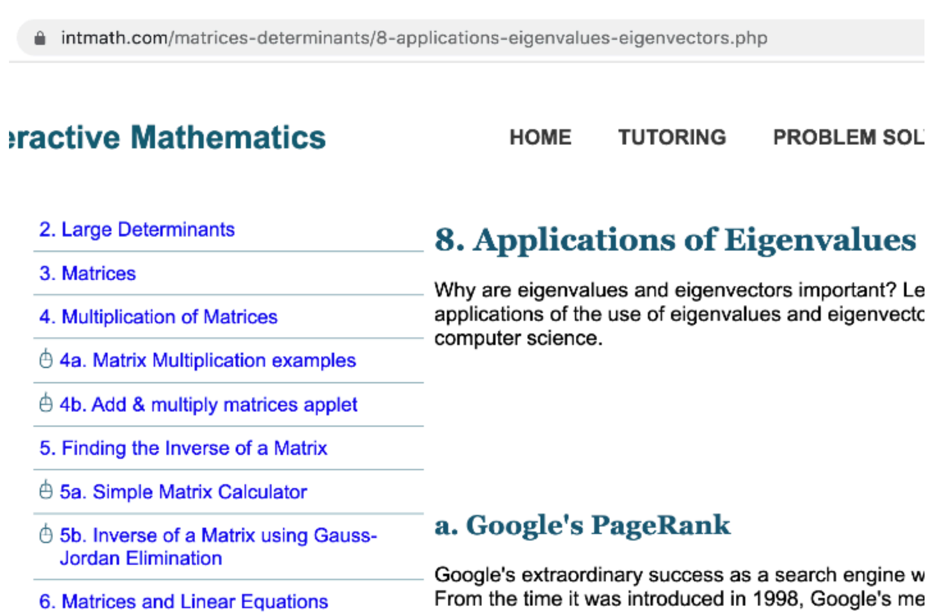
Search [搜索]

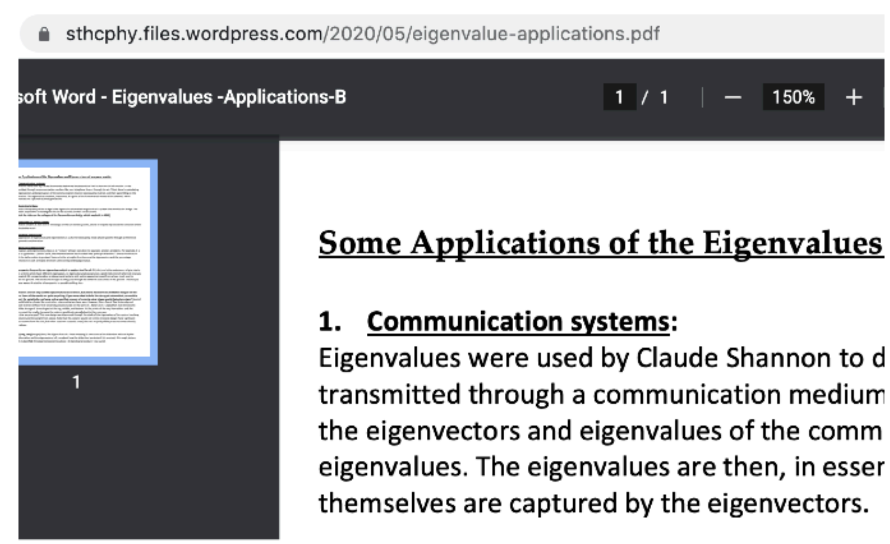
Search (key IT problem): pick good webpages

3 key IT problems:
Search
Recommendation,
Advertisement
[搜推广]

e.g. Many many webpages related to “application of eigenvalues”



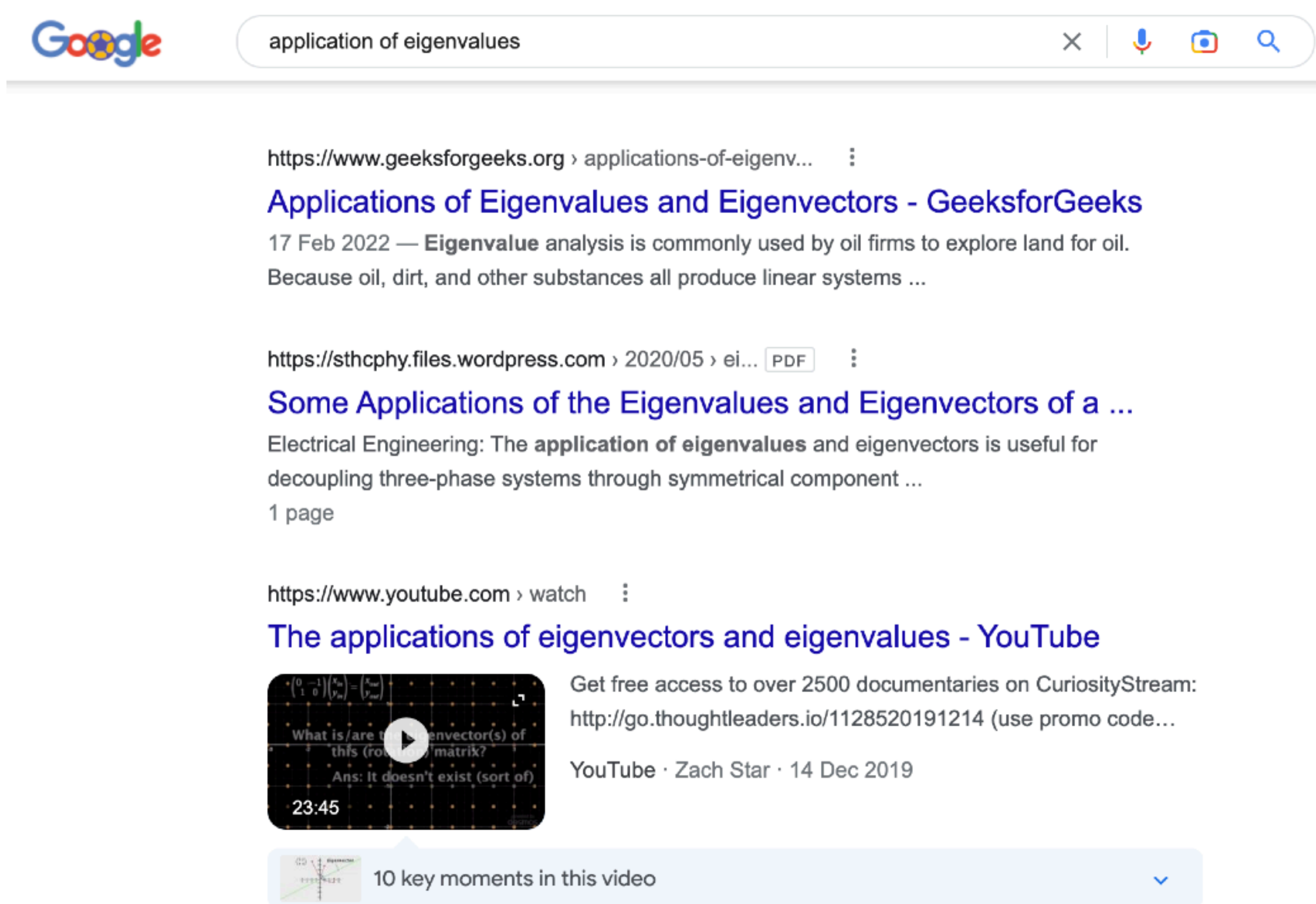




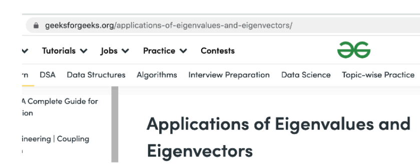
PageRank: Algorithm for Search

Search (key IT problem): viewed as ranking items

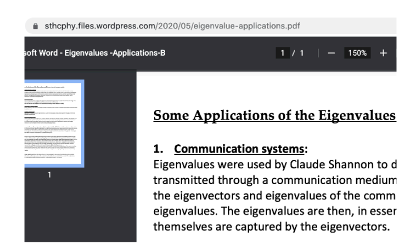
PageRank: algorithm to rank webpages



Rank



1



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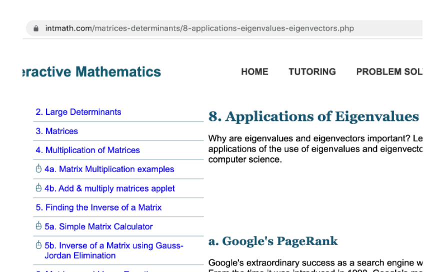


3

...

Some Thoughts

General advice: before reading any algorithm, think first:

What would you do?

Yiming Zhang 张一鸣 (ByteDance): excellent example.
Solve recommendation in his own way.

Rank by importance of webpages.

Correct but useless!

Some Thoughts

General advice: before reading any algorithm, think first:

What would you do?

Rank by importance of webpages.

How to compute "importance"?

1) Authority? [权威]

2) Number of visitors? [访问量]

3) Number of links to the webpage?

—degree of a graph

True hotspot (linked to real webpages)

Fake hotspot (with "fake fans")

New Thought

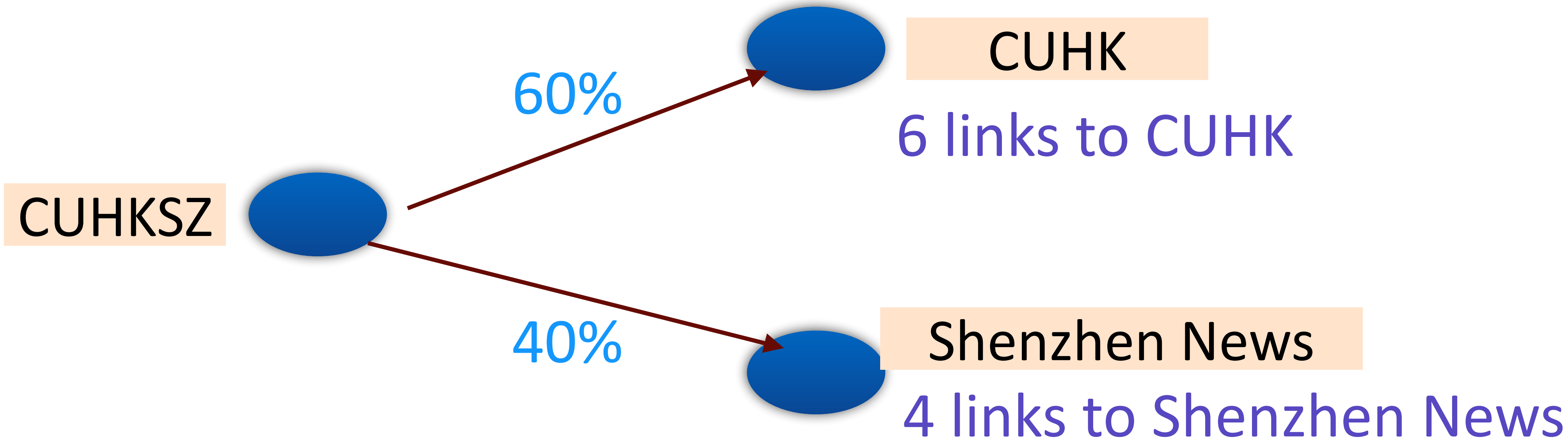
What did Page and Burin do in 1998?

How to compute "importance" of webpages?

1) View webpages as a graph.
—If webpage 1 has a link to webpage 2, then add directed edge (1, 2).

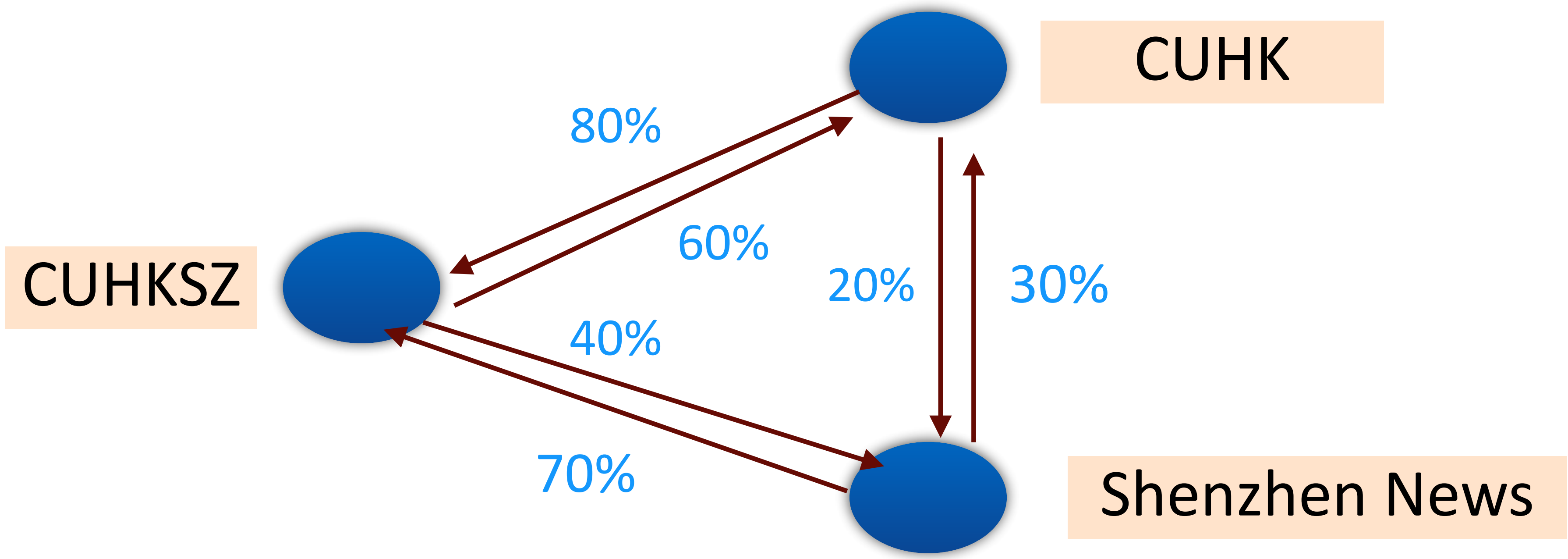
2) Estimate # of visitors.

Key idea: mimic browsing.



Browsing Dynamics

Assume there are 1000 visitors on each webpage at minute 1.
How many on each webpage at minute 2?



Visitors to CUHKSZ	x_t
Visitors to CUHK	y_t
Visitors to SZ News	z_t

$$x_{t+1} = \quad x_t + \quad y_t + \quad z_t$$

$$y_{t+1} = \quad x_t + \quad y_t + \quad z_t$$

$$z_{t+1} = \quad x_t + \quad y_t + \quad z_t$$

Model for General Graph

Setting: n webpages.

Each webpage may have links to other webpages.

At time t, there are $v_1(t)$, $v_2(t)$, …, $v_n(t)$ visitors at page $1, 2, \ldots, n$.

Assumptions:

If there are m pages at a webpage j,

then a visitor at webpage j will click one of the m pages randomly.

Denote $a_{j,k}$ as the probability of a visitor at webpage j to visit webpage k .

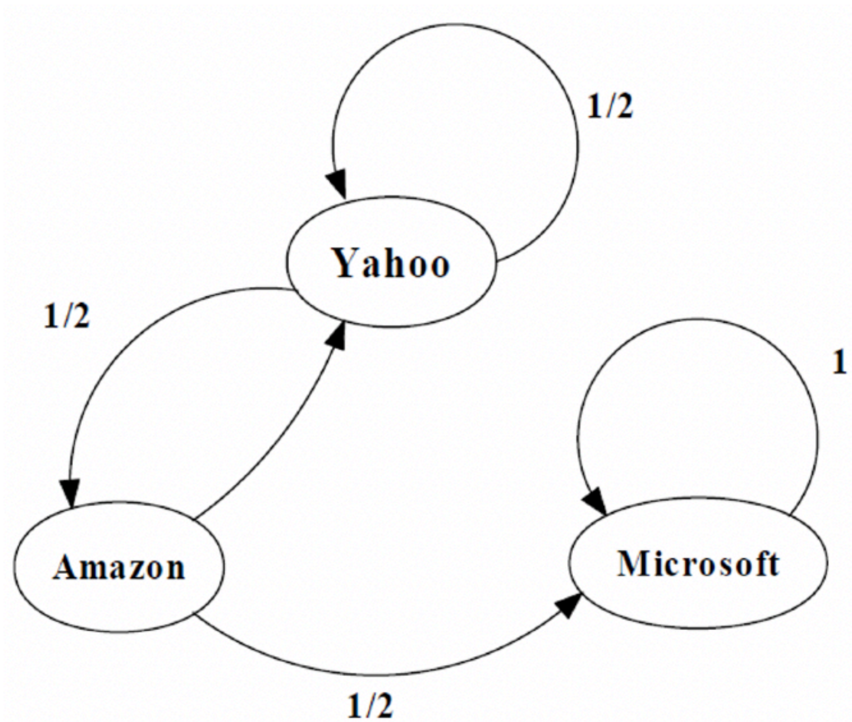
$A = (a_{jk})_{1 \leq j,k \leq n}$

Math model:

$$\begin{bmatrix} v_1(t+1) \\ v_1(t+1) \\ \vdots \\ v_n(t+1) \end{bmatrix} = A \begin{bmatrix} v_1(t) \\ v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

Algorithm: Compute $\mathbf{v}(\infty) \triangleq \lim_{t \to \infty} \mathbf{v}(t)$, and rank webpages by entries of $\mathbf{v}_\infty$.

An Example

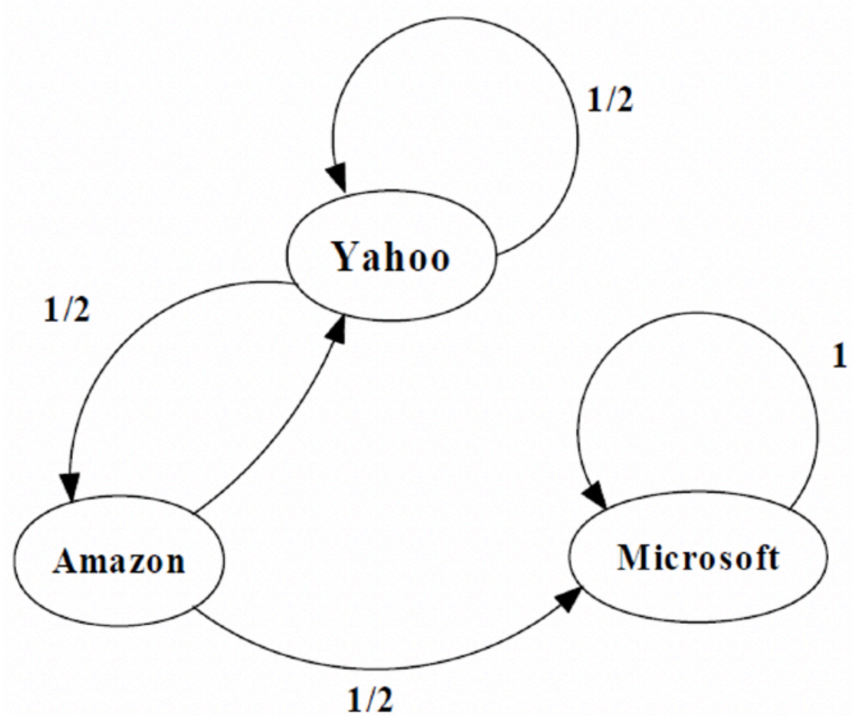


$$M = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \text{yahoo} \\ \text{Amazon} \\ \text{Microsoft} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Time 1: $$\begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

An Example



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Time 1: $\begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

Time 2: $\begin{bmatrix} 1/4 \\ 1/6 \\ 7/12 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix}$

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \quad \begin{bmatrix} 1/4 \\ 1/6 \\ 7/12 \end{bmatrix} \quad \begin{bmatrix} 5/24 \\ 1/8 \\ 2/3 \end{bmatrix} \quad \begin{bmatrix} 1/6 \\ 5/48 \\ 35/48 \end{bmatrix} \quad \ldots$$

Solution

Math model:

$$\begin{bmatrix} v_1(t+1) \\ v_1(t+1) \\ \vdots \\ v_n(t+1) \end{bmatrix} = A \begin{bmatrix} v_1(t) \\ v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

$$\mathbf{v}(t+1) = A\mathbf{v}(t)$$

Algorithm: Compute $\mathbf{v}(\infty) \triangleq \lim_{t\to\infty} \mathbf{v}(t)$, and rank webpages by entries of $\mathbf{v}_\infty$.

Observation:

$$\mathbf{v}(\infty) = A\mathbf{v}(\infty)$$

Thus $\mathbf{v}(\infty)$ is ________________________________