Lecture 25

Singular Value I: Forms and Properties of SVD

Instructor: Ruoyu Sun

香港中文大學（深圳） | 数据科学学院
The Chinese University of Hong Kong, Shenzhen | School of Data Science

Roadmap of MATH2041

Segment 1

Solving linear systems (Lec 3-15);

Solving least squares problem (Lect 16,17)

Segment 2

[Lec 18-21]: Two relatively independent parts:

—Determinant. [Important tool!]

—Linear transformation. [Advanced math perspective of matrix]

Segment 3

Next: Lec 22-27: Eigenvalues and related.

Schedule:

—Eigenvalues. Lec 22-24

—**Singular values**. Lec 25-26

—Quadratic forms. Lec 27.

Outline of SVD

1) **Forms** of SVD.

2) **Computation** of SVD.

3) **Properties** of SVD.

4) PCA and image/data compression (if time permitting)

Review

Eigenvalues and Eigenvectors

Definition 21.1 (Eigenvalues and Eigenvectors)

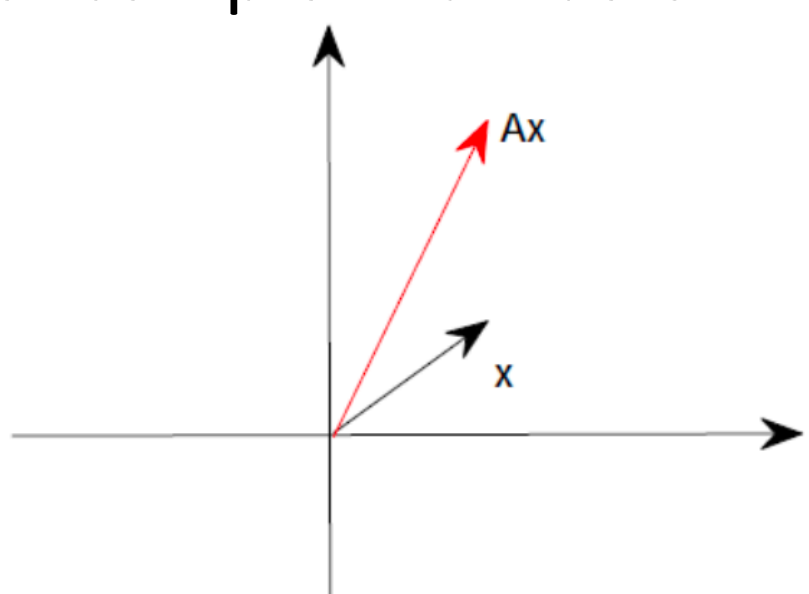
Let $A \in \mathbb{C}^{n \times n}$ be a square matrix.

If there exists a scalar λ ($\in \mathbb{R}$ or $\mathbb{C}$) and a **nonzero** vector x such that

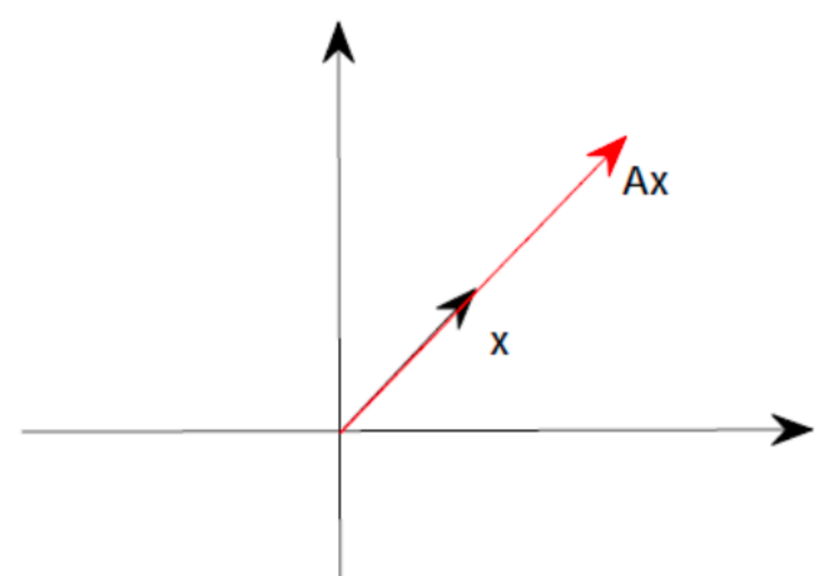
$$Ax = \lambda x,$$

then λ is called a (real or complex) **eigenvalue** and x is called an **eigenvector** with respect to (or associated with; corresponding to) λ.

$\mathbb{C}$:= set of complex numbers



x is **not** an eigenvector



x is an eigenvector

General Procedure to Find Eigenvalues/Eigenvectors

Step 1: Solve $\det(A - \lambda I) = 0$ and get n roots $\lambda_1, \ldots, \lambda_n$

Solving single-var polynomial.

Step 2: For each λ_i, find the eigenspace $\mathrm{Null}(\lambda_i I - A)$

Solving up to n linear systems.

Any nonzero vector in $\mathrm{Null}(\lambda_i I - A)$ is an eigenvector corresponding to λ_i

$$\mathrm{Null}(\lambda_i I - A) = \left\{ \begin{array}{l} \text{eigenvectors of } A \text{ with} \\ \text{respect to the eigenvalue } \lambda_i \end{array} \right\} \bigcup \ \{0\}$$

Spectral Theorem

Theorem 23.1 [Spectral Theorem]

Any real symmetric matrix A can be written as

$$A = VDV^\top = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^\top . \qquad (*)$$

where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a real diagonal matrix,

$V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is a real orthogonal matrix.

(*): Eigenvalue decomposition (EVD) or eigendecomposition of A.

Judgement Question

Judgement: Any real square matrix A can be written as $A = SDS^\top$ where D is a diagonal matrix, and S is an orthogonal matrix.

Judgement: If $A, B \in \mathbb{R}^{n \times n}$ are similar, then A and B have the same multiset of eigenvalues.

Judgement: If $A, B \in \mathbb{R}^{n \times n}$ have the same multiset of eigenvalues, then they are similar.

Judgement: A real square matrix is similar to a real diagonal matrix.

Judgement: If $A, B \in \mathbb{R}^{n \times n}$ are similar, then tr(A) = tr(B).

Part I SVD

Judgement Questions

Q1: Any rectangular matrix has at least one complex eigenvalue.

Q2: Eigenvectors of an $n \times n$ matrix can form a basis $\mathbb{C}^n$.

Q3: Eigenvectors of an $n \times n$ matrix corresponding to different eigenvalues are orthogonal.

Facts About Eigenvalues

Fact 1: Eigenvalues are defined for _______ matrices, NOT _______ matrices.

Fact 2: The eigenvectors of a general matrix ____________ form a basis of the whole space.

Fact 3: The eigenvectors of a general matrix ____________ orthogonal.

Textbook: Singular vectors solve these problems in a perfect way. SVD is a highlight of linear algebra.

Full SVD

Rectangular diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ is a matrix of the form

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \dots & 0 \\ 0 & 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

Denote as $\Sigma = \text{rec-diag}(\sigma_1, \dots, \sigma_{\min(m,n)})_{m\times n}$

Theorem 25.1 [Full SVD]

Any matrix $A \in \mathbb{R}^{m\times n}$ can be written as

$$A = U\Sigma V^\top = \sum_{j=1}^{\min(m,n)} \sigma_j \mathbf{u}_j \mathbf{v}_j^\top, \qquad (*)$$

where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are two orthogonal matrices,

$\Sigma = \text{rec-diag}(\sigma_1, \dots, \sigma_{\min(m,n)})_{m\times n} \in \mathbb{R}^{m\times n}$ is a rectangular diagonal matrix.

Full SVD

Rectangular diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ is a matrix of the form

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \dots & 0 \\ 0 & 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

Denote as $\Sigma = \text{rec-diag}(\sigma_1, \dots, \sigma_{\min(m,n)})_{m\times n}$

Theorem 25.1 [Full SVD]

Any matrix $A \in \mathbb{R}^{m\times n}$ can be written as

$$A = U\Sigma V^\top = \sum_{j=1}^{\min(m,n)} \sigma_j \mathbf{u}_j \mathbf{v}_j^\top, \quad (*)$$

where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are two orthogonal matrices,

$\Sigma = \text{rec-diag}(\sigma_1, \dots, \sigma_{\min(m,n)})_{m\times n} \in \mathbb{R}_+^{m\times n}$ is a rectangular diagonal matrix.

Corollary 25.1:

If (*) holds, then columns of U are eigenvectors of $AA^\top \in \mathbb{R}^{m\times m}$, called left singular vectors of A;

columns of V are eigenvectors of $A^\top A \in \mathbb{R}^{n\times n}$, called right singular vectors of A;

σ_j's are called singular values of A, and $\sigma_j = \sqrt{\lambda_j}$ where $\lambda_1, \dots, \lambda_{\min(m,n)}$ are eigenvalues of both $AA^\top$ and $A^\top A$.

Proof Preparation: Two Lemmas

Lemma 25.1 [eigenvectors and singular vectors]

Suppose $A^\top A v = \lambda v$, where $\lambda \neq 0, v \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$, then $Av \neq \mathbf{0}_m$.

Suppose $A^\top A v = \mathbf{0}_n$, then $Av = \mathbf{0}_m$.

Lemma 25.2 [singular-equation-pairs in matrix form]

Suppose $A \in \mathbb{R}^{m \times n}, Av_1 = \sigma_1 u_1, \ldots, Av_k = \sigma_k u_k,$ where $v_j \in \mathbb{R}^{n \times 1}, u_j \in \mathbb{R}^{m \times 1}, \sigma_j \in \mathbb{R}$.

Then $AV = UD$, (*)

where $D = \text{diag}(\sigma_1, \ldots, \sigma_k)$ is a diagonal matrix, $V = [v_1, \ldots, v_k], U = [u_1, \ldots, u_k]$.

Proof Analysis (II): What's Next after AV = UD?

Lemma 25.1 [eigenvectors and singular vectors]

Suppose $A^\top Av = \lambda v$, where $\lambda \neq 0, v \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$, then $Av \neq \mathbf{0}_m$.

Suppose $A^\top Av = \mathbf{0}_n$, where $v \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$, then $Av = \mathbf{0}_m$.

From Lemma 25.1, we can construct a few equations

Lemma 25.2 [singular-equation-pairs in matrix form]

Suppose $A \in \mathbb{R}^{m\times n}, Av_1 = \sigma_1 u_1, \ldots, Av_k = \sigma_k u_k,$ where $v_j \in \mathbb{R}^{n\times 1}, u_j \in \mathbb{R}^{m\times 1}, \sigma_j \in \mathbb{R}$.

Then $AV = UD$, (*)

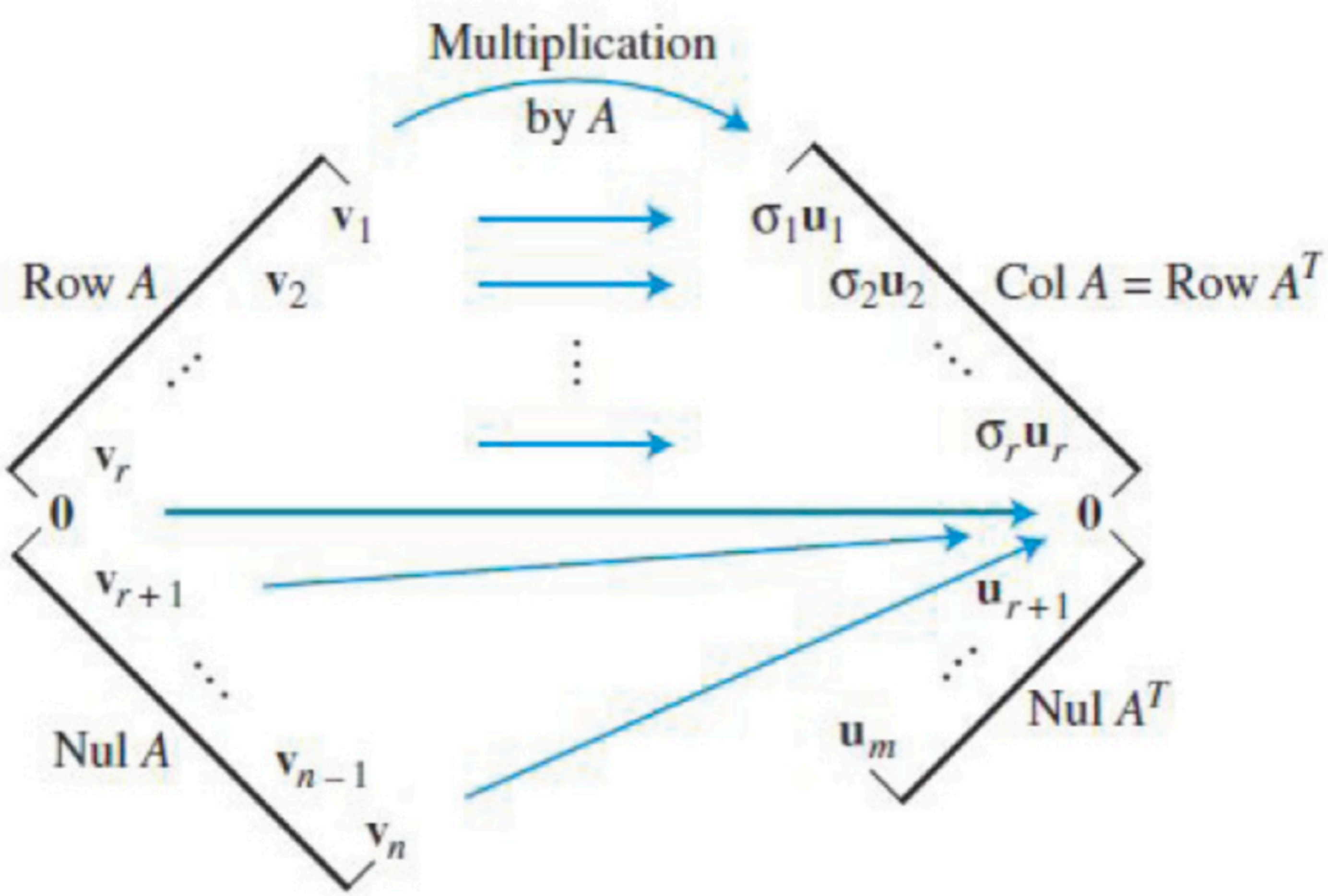
where $D = \text{diag}(\sigma_1, \ldots, \sigma_k)$ is a diagonal matrix, $V = [v_1, \ldots, v_k], U = [u_1, \ldots, u_k]$.

From AV = UD, can we get an expression of A?

Proof Analysis (III): Completing Basis

V = $[v_1, \ldots, v_r]$, U = $[u_1, \ldots, u_r]$.

From AV = UD, can we get an expression of A?



Wrap-up: Proof Sketch of full SVD Theorem

Assume $m \geq n$ (for case $m > n$, consider $\tilde{A} = A^\top$).

Step 1: Suppose $A^\top A$ has eigenvalues $\lambda_1 \geq \ldots \geq \lambda_r > 0 = \lambda_{r+1} = \ldots = \lambda_n \quad (1)$.

By spectral theorem, there exists orthonormal eigenbasis of $A^\top A$: $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$,

s.t. $A^\top A \mathbf{v}_j = \lambda_j \mathbf{v}_j$.

Step 2: We have $A\mathbf{v}_n = 0$ because $\|A\mathbf{v}_n\|^2 = \mathbf{v}_n^\top A^\top A \mathbf{v}_n = \mathbf{v}_n^\top(\lambda_n \mathbf{v}_n) = 0.$

[Essentially proved: $A^\top A\mathbf{v} = 0 \Rightarrow \mathbf{Av} = 0.$]

By same reasoning, $A\mathbf{v}_{r+1} = \ldots = A\mathbf{v}_n = 0 \quad (2).$

Step 3: Let $\sqrt{\lambda}_1 \mathbf{u}_1 = A\mathbf{v}_1, \quad \ldots, \quad \sqrt{\lambda}_r \mathbf{u}_r = A\mathbf{v}_r$ **(3)**

Easy to show: they are unit-norm, and orthogonal.

Let $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^m$ be orthonormal basis of the span $(\{\mathbf{u}_1, \ldots, \mathbf{u}_r\})^\perp$.

By (1),(2): $\sqrt{\lambda}_{r+1} \mathbf{u}_{r+1} = 0 = A\mathbf{v}_1, \quad \ldots, \quad \sqrt{\lambda}_n \mathbf{u}_n = 0 = A\mathbf{v}_n \quad (4)$

Step 4: Verify $AV = U\Sigma$ using (3),(4). Since $V^\top V = I_n$, get $A = U\Sigma V^\top$.

Remark: Directly put (3), (4) in matrix form, can get reduced SVD, not full SVD.

Compact SVD

Theorem 25.2 [Compact SVD and vector form]

Any matrix $A \in \mathbb{R}^{m\times n}$ with rank r can be written as

$$A = U_r \Sigma_r V_r^\top = \sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^\top, \quad (*)$$

where $U_r = [\mathbf{u}_1, \ldots, \mathbf{u}_r] \in \mathbb{R}^{m\times r}$ has orthogonal unit-norm columns,

$V_r = [\mathbf{v}_1, \ldots, \mathbf{v}_r] \in \mathbb{R}^{n\times r}$ has orthogonal unit-norm columns,

$\Sigma_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r) \in \mathbb{R}_+^{r\times r}$ is a diagonal matrix.

Proof: Direct Corollary of Theorem 25.1

Number of singular values

For $A \in \mathbb{R}^{m\times n}$, $A = U\Sigma V^T$ and rank(A)=r, one has
$\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$ are the eigenvalues of $A^T A$.
$\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_m = 0$ are the eigenvalues of AA^T.

If $m \geq n$, singular values are

If $m \leq n$, singular values are

In summary, the number of singular values is ____________.

Part II Computation of SVD

Example 1: Compute SVD

Problem 1: Find the SVD of the matrix A = $\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.

Solution:

Step 1: Find eigenvalues of A'A.

$$A^{\mathrm{T}}A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \qquad AA^{\mathrm{T}} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

Eigenvalues of $A^T A$ are 45 and 5.

Problem 1: Step 1 and Step 2

Problem 1: Find the SVD of the matrix A = $\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.

Solution:

Step 1: Find eigenvalues of AA' and A'A.

$$A^{\mathrm{T}}A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \qquad AA^{\mathrm{T}} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

Eigenvalues of A^TA are 45 and 5.

Step 2: Find (unit) eigenvectors of A'A, i.e. right singular vectors of A.

$$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{45}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{5}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Right singular vectors $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Problem 1: Step 3 and Remark

Step 3: Find left singular vectors of A. **Left singular vectors** $\boldsymbol{u}_i = \dfrac{A\boldsymbol{v}_i}{\sigma_i}$

Now compute $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ which will be $\sigma_1\boldsymbol{u}_1 = \sqrt{45}\,\boldsymbol{u}_1$ and $\sigma_2\boldsymbol{u}_2 = \sqrt{5}\,\boldsymbol{u}_2$:

$$A\boldsymbol{v}_1 = \frac{3}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} = \sqrt{45}\frac{\mathbf{1}}{\sqrt{\mathbf{10}}}\begin{bmatrix}\mathbf{1}\\\mathbf{3}\end{bmatrix} = \sigma_1\boldsymbol{u}_1$$

$$A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-3\\1\end{bmatrix} = \sqrt{5}\,\frac{\mathbf{1}}{\sqrt{\mathbf{10}}}\begin{bmatrix}\mathbf{-3}\\\mathbf{1}\end{bmatrix} = \sigma_2\boldsymbol{u}_2$$

Conclusion: The SVD of A is $A = U\Sigma V^\top$ where

$$\boldsymbol{U} = \frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix}\sqrt{45} & \\ & \sqrt{5}\end{bmatrix} \qquad \boldsymbol{V} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

Remark: Verify your SVD by checking ___________________________

Example 2: Compute SVD

Problem 2: Find the (full) SVD of the matrix A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Solution:

Step 1: Find eigenvalues of A'A (which is 2 by 2, where 2 = min(m,n))

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Eigenvalues of $A^T A$ are 4 and 0.

Example 2: Compute SVD

Problem 1: Find the SVD of the matrix A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Solution:

Step 1: Find eigenvalues of A'A (which is 2 by 2, where 2 = min(m,n))

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Eigenvalues of $A^T A$ are 4 and 0.

Step 2: Find (unit) eigenvectors of A'A, i.e. right singular vectors of A.

The unit eigevector w.r.t. $\lambda_1 = 4$ is $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The unit eigevector w.r.t. $\lambda_2 = 0$ is $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Problem 1: Step 3 and Remark

Step 3: Find left singular vectors of A. **Left singular vectors** $u_i = \dfrac{Av_i}{\sigma_i}$

Now $\sigma_1 = \sqrt{\lambda_1} = 2$ so

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\\ 0 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

An orthonormal basis for Null(A^T)=Null(AA^T) is

$$\{\mathbf{u}_2, \mathbf{u}_3\} = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}, \quad \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right\}$$

Conclusion: The full SVD of A is

$$\begin{bmatrix}1 & 1\\ 1 & 1\\ 0 & 0\end{bmatrix} = A = U\Sigma V^T = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Remark: Verify your SVD by checking ________________________

Part III Properties of SVD

Four Fundamental Subspaces

$$A = [u_1, \cdots, u_r, u_{r+1}, \cdots, u_m] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n \end{bmatrix}.$$

$R(A)$ column space : dim r, orthonormal basis $u_1, u_2, \cdots, u_r$.

null space : dim $n-r$, — — basis $v_{r+1}, \cdots, v_n$.

row space : dim r, — — basis $v_1, \cdots, v_r$.

leftnull space : dim $m-r$, — — — basis $u_{r+1}, \cdots, u_m$.

Row Space and Columns Space

Lemma 25.1 Suppose $A = \sum_{j=1}^{k} \mathbf{x}_j \mathbf{y}_j^\top$,

where

$\mathbf{x}_j's$ are linearly independent, $\mathbf{y}_j's$ are linearly independent.

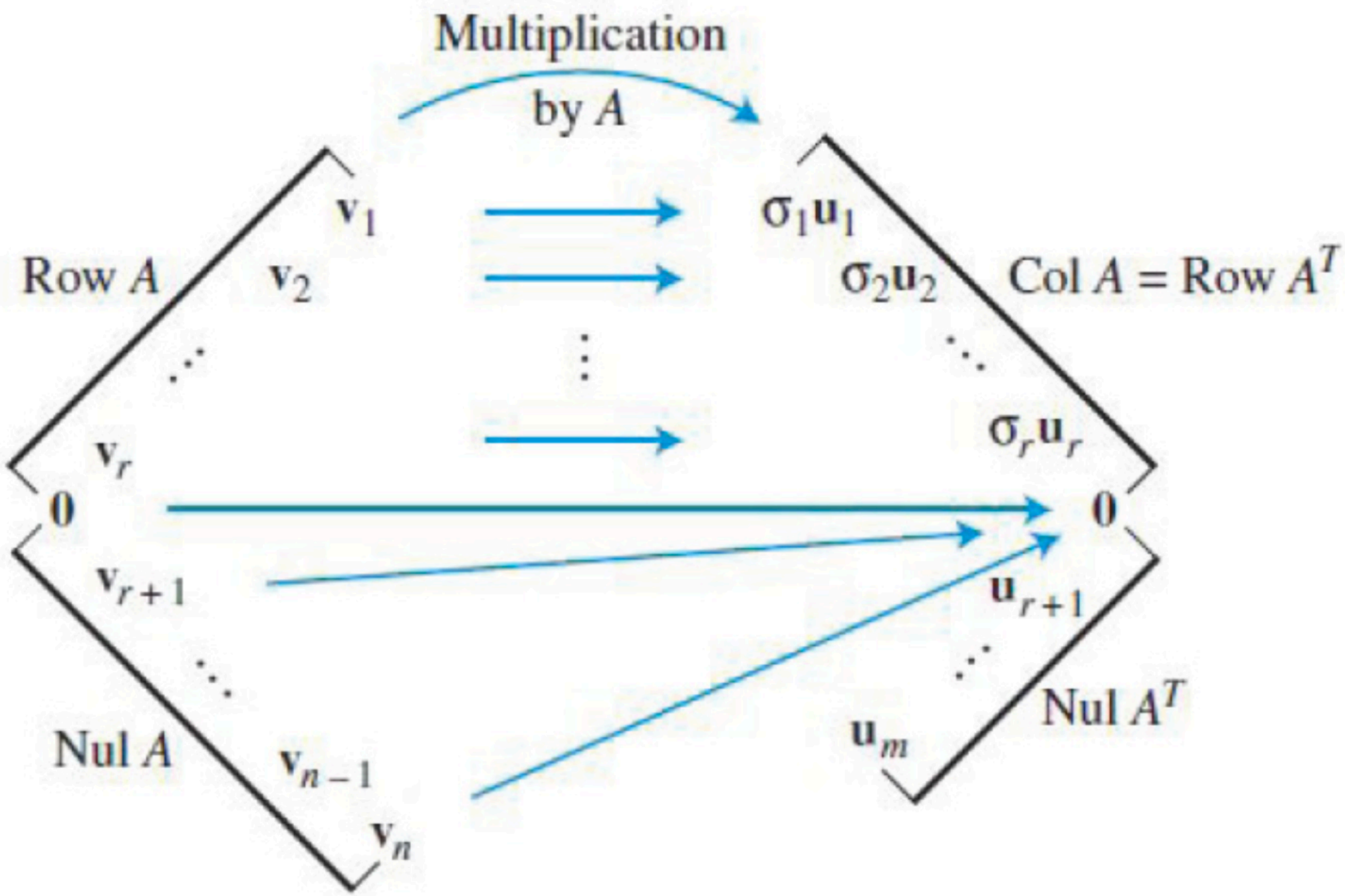
Then Row(A) = span$\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$, C(A) = span$\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

Lemma 25.2 (equivalent) Suppose $A = XY^\top$,

where $X \in \mathbb{R}^{m \times r}$, $Y \in \mathbb{R}^{n \times r}$ have full rank.

Then C(A) = $C(X)$, Row(A) = Row$(Y^\top)$.

Four Fundamental Subspaces



Rank = # of Nonzero Singular Values

rank(A) = # of nonzero singular values (counting multiplicity)

rank(A) ________ # of nonzero eigenvalues.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues are ___________,

# of nonzero eigenvalue is _______

rank(A) = ______.

Judgement

The number of singular values is the same as the rank of the matrix.

The number of singular values of an $m \times n$ matrix is m or n.

The rank of an $m \times n$ matrix is the number of nonzero eigenvalues.

Part IV Geometry

Facts About Eigenvalues

In geometrical interpretation of eigenvectors, we view the matrix as a linear transformation between ____and ______.

Why?

Two geometrical views:

—1st view: “Unchanged” during “change”; fixed point.
This motivation only applies to “self-transformation”.

—2nd view: “Tiles” —> similar tiles with stretching.

Two-Space Transformation: Standard Basis

Consider $A \in \mathbb{R}^{3\times 2}$.

A can be viewed as a linear transformation: $\mathbb{R}^2 \to \mathbb{R}^3$

Geometrical view:

Changing tiles: ________________ to ______________________

Two-Space Transformation: Arbitrary Basis

Consider an arbitrary basis.

Changing tiles: _______________ to _______________________

Two-Space Transformation: Arbitrary Basis

Wish: Find basis s.t. shape of "tiles" do NOT change (too much).

Changing tiles: ______________ to ______________________

Mathematically:

Find orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$s.t. $\{A\mathbf{v}_1, A\mathbf{v}_2\}$ is _______________

Summary Today (Write Your Own)

One sentence summary:

Detailed summary:

Summary Today (Write Your Own)

One sentence summary:

We learn SVD today.

Detailed summary:

1) SVD is a decomposition of any real matrix A (can be rectangular) s.t.

$$A = UDV^{\top},$$

where

1b) Compact SVD:

2) Relation of singular values/vectors and eigenvalues/eigenvectors:

3) Number of singular values is _______.

4) Steps of computing SVD: