Lecture 03

Systems of Linear Equations I: Forms and Elimination

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Recall

In the last lectures …

- Definition of norm and dot (inner product)
- Calculation of vector norms and inner products
- Real-world examples

Today

Today … System of **Linear Equations!**

After this lecture, you should be able to

1. Write the 4 forms of **systems of linear equations**

2. Write the various forms of **matrix-vector product**

2. Solve a linear system by **Gaussian elimination**

Reading Material: Logic and Proof

Difficulties in Linear Algebra, partially due to…

lack of training of proving.

—or simply, lack of LOGIC.

Material to check:

https://www.math.toronto.edu/preparing-for-calculus/3_logic/we_2_if_then.html

logic is about telling a complete story

Part I System of Linear Equations

Linear System of Equations: Preliminary School Example

Problem (Chicken-Rabit Problem 鸡兔同笼)

There are 35 heads and 94 feet in a cage.
How many chickens and how many rabbits are there?

Assumption: Each chicken has 1 head and 2 feet, and each rabbit has 1 head and 4 feet.

$$\begin{cases} x + y = 35 \\ 2x + 4y = 94 \end{cases}$$

二元一次方程组
two-variable
order-1 system of equations

Linear Equations

Definition (Linear Equations)

A **linear equation** with n unknowns is the equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

or $\vec{a} \cdot \vec{x} = b$

where $a_1, a_2, \ldots, a_n, b$ are real numbers and $x_1, x_2, \ldots, x_n$ are variables

$\vec{a} = [a_1, \ldots, a_n]$, $\vec{x} = [x_1, \ldots, x_n)$

coefficient variable

Exercise

Exercise (Linear Equations)

Are the following linear equations?

1. $-x_1 + 4x_4 = 2x_2 + 3x_3$ yes

x_1, x_2, x_3, x_4 are variables

Exercise

Exercise (Linear Equations)

Are the following linear equations?

1. $-x_1 + 4x_4 = 2x_2 + 3x_3$ $\quad$ x_1, x_2, x_3, x_4 are variables

2. $-x_1 x_4 = 2x_2 + 3x_3$ $\quad$ no $\quad$ x_1, x_2, x_3, x_4 are variables

$x_1 x_4 = 2x_2$, where x_2, x_4 are variables

Then it's a linear equation.

Exercise

Exercise (Linear Equations)

Are the following linear equations?

1. $-x_1 + 4x_4 = 2x_2 + 3x_3$ x_1, x_2, x_3, x_4 are variables
2. $-x_1x_4 = 2x_2 + 3x_3$ x_1, x_2, x_3, x_4 are variables
3. $a_1x_1 + a_2x_2 = b$ no x_1, x_2, a_1, a_2, b are variables

$x+y=5$. 1×2

System of Linear Equations

$\begin{cases} x+y=1 \\ 2x+4y=2 \end{cases}$ 2×2

Definition (System of Linear Equations)

An $m \times n$ **system of linear equations** is a *collection* of m linear equations with n variables

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where all a_{ij} and b_i are real numbers and $x_1, x_2, \ldots, x_n$ are variables

(A system of linear equations is also called a **linear system**)

Exercise

Exercise (Linear Equations)

Is the following a linear system?

$$\begin{cases} -a_{11} + 4a_{12} = 2a_{13} + 3a_{14} \\ 5a_{11} + 3a_{12} = a_{13} + 3a_{14} \end{cases}$$

$a_{11}, a_{12}, a_{13}, a_{14}$ are variables

Exercise

Exercise (Linear Equations)

Is the following a linear system?

$$-a_{11} + 4a_{12} = 2a_{13} + 3a_{14}$$

$$5a_{11} + 3a_{12} = a_{13} + 3a_{14}$$

$a_{11}, a_{12}, a_{13}, a_{14}$ are variables

It is critical to know what are the **variables (unknowns)**!

(such as the weight vectors in our movie preference example)

What is New in this Course?

You learned these in middle school (or even primary school).

What more can we study? What will be new?

1. Yes, you can solve system of equations in 2 variables.
   What about 5 variables?
   What about 100 variables?

   for computers

   What is a general method to solve an any-variable system?

2. Theory: Does your method always work?
   Can you prove it? etc.

Part II Matrix-Vector Product & Four Forms of Linear Systems

Textbook Sec. 1.3 (only first half) and Sec. 2.1

Row-Vector Form of System

Scalar form (F1)

$$\begin{cases} 10x_1 + 8x_2 + 5x_3 = 8.5 \\ 8x_1 + 5x_2 + 10x_3 = 6.5 \\ 5x_1 + 4x_2 + 6x_3 = 4.9 \end{cases}$$

or $[10, 8.5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 8.5$

$[10, 8, 5] \cdot [x_1, x_2, x_3] = 8.5$

$[8, 5, 10] \cdot [x_1, x_2, x_3] = 6.5$ (F2)

$[5, 4, 6] \cdot [\; - \;] = 4.9$

Rewrite as 3 equations: vector form

dot product

Unknowns

$$\begin{matrix} (10 \quad 8 \quad 5) \\ (8 \quad 5 \quad 10) \\ (5 \quad 4 \quad 6) \end{matrix} \bullet (w_1 \quad w_2 \quad w_3) = \begin{pmatrix} 8.5 \\ 6.5 \\ 4.9 \end{pmatrix}$$

Column-Vector Form

scalar

$$\begin{cases} 10x_1 + 8x_2 + 5x_3 = 8.5 \\ 8x_1 + 5x_2 + 10x_3 = 6.5 \\ 5x_1 + 4x_2 + 6x_3 = 4.9 \end{cases} \quad (F1)$$

Another way of writing the equations: using v_1, v_2, v_3.

$v_1 = \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$.

not easy

easy

Column-vector form

Equivalently

$$\begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix} x_3 = \begin{bmatrix} 8.5 \\ 6.5 \\ 4.9 \end{bmatrix} \quad (F3)$$

Row-form: Simplify by Product

Motivation: Make (F1), (F2), (F3) simple.

$$\begin{cases} x_1 + x_2 = 14 \\ 2x_1 + 4x_2 = 36 \end{cases} \xrightarrow{\text{row-vec}} \underbrace{\begin{cases} [1,1] \cdot [x_1, x_2] = 14 \\ [2,4] \cdot [x_1, x_2] = 36 \end{cases}}_{\text{define it as } A \cdot x}$$

used twice

Motivation Make it simple.

$$A := \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 14 \\ 36 \end{bmatrix}.$$

Define $A \cdot x = \begin{bmatrix} [1,1] \cdot [x_1, x_2] \\ [2,4] \cdot [x_1, x_2] \end{bmatrix}$.

then the system becomes $Ax = b$.

Introducing Matrix-Vector Product

Motivation: Make (F1), (F2), (F3) simple. Define $A=\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}$, $b=\begin{bmatrix}16\\94\end{bmatrix}$, $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$.

(F1) $\begin{cases} x_1+x_2=16 \\ 2x_1+4x_2=94 \end{cases}$ $\xrightarrow{\text{want}}$ $Ax=b$

need to define $Ax=\begin{bmatrix}x_1+x_2\\ 2x_1+4x_2\end{bmatrix}$

(F2) $\begin{cases} [1,1]\cdot[x_1,x_2]=16 \\ [2,4]\cdot[x_1,x_2]=94 \end{cases}$ $\xrightarrow{\text{want}}$ $Ax=b$

need to define $Ax=\begin{bmatrix}[1,1]\cdot[x_1,x_2]\\ [2,4]\cdot[x_1,x_2]\end{bmatrix}$

(F3) $\begin{bmatrix}1\\2\end{bmatrix}x_1+\begin{bmatrix}1\\4\end{bmatrix}x_2=\begin{bmatrix}16\\94\end{bmatrix}$ $\xrightarrow{\text{want}}$ $Ax=b$

need to define $Ax=\begin{bmatrix}1\\2\end{bmatrix}x_1+\begin{bmatrix}1\\4\end{bmatrix}x_2$

Remark: Previous two pages: $(F1)\Leftrightarrow(F2)$. $(F1)\Leftrightarrow(F3)$.

$\Rightarrow$ $(F2)\Leftrightarrow(F3)$ ✓ NOT straightforward why this holds!

Introduce matrix so this relation is easier to memorize.

Three Definitions of Matrix-Vector Product

Ignore equations. Summarize the last page.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

<u>Def-1.</u> $Ax = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$

<u>Def-2</u> $Ax = \begin{bmatrix} [1,1] \cdot [x_1, x_2] \\ [2,4] \cdot [x_1, x_2] \end{bmatrix}$

<u>Def-3</u> $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2$

<u>Claim</u> Three definitions are equivalent.

[easy to check: (F1) $\Leftrightarrow$ (F2); (F1) $\Leftrightarrow$ (F3)]

Next we extend these definitions to general matrix, vector.

Matrix Definition

Definition 4.1 (Matrix)

An $m \times n$ **matrix** A is a rectangular array of numbers with m *rows* and n *columns* in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m \times n} \in \mathbb{R}^{m \times n}.$$

where all a_{ij} are scalars.

wrong: dim is mn; e.g. dim is 200. Correct: dim is 10x20

Def 4.2 (Dimension):

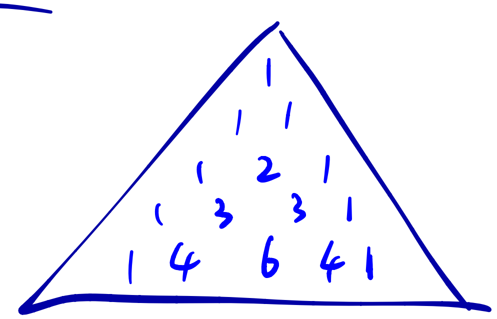
For an $m \times n$ **matrix,** the dimension of A is $m \times n$ (reads “m by n”).
For an $m \times 1$ vector, the dimension of the vector is m, or $m \times 1$.

Matrix

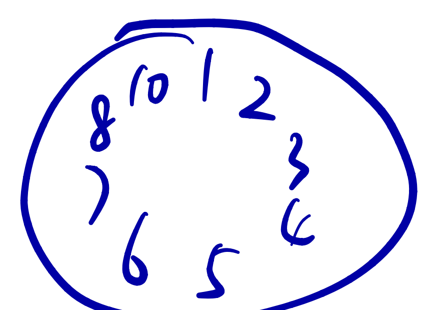
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -3 & 4.7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{pmatrix}$$

Non-example. [collection of number may NOT be matrix]



Yang-Hui Triangle



clock

Matrix Conventions (Directly Copied from Lecture 4)

- For a matrix A, a_{ij} is called the (i, j)-th entry (element) of A

  (2,3)-th entry of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is 6

- Matrices are denoted by $A, B, C, \ldots$

- When $m = n$, A is called a *square matrix;* a *rectangular matrix o.w.*

  方阵

- When all entries are zeros A is called a *zero matrix*

  (similar to zero vector) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- When $m = 1$, A is a row vector $[1, 2, 3]$

- When $n = 1$, A is a column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- When $m = n = 1$, A can be considered as a scalar $[1.57]$

  In math, 1x1 matrix = scalar; In coding, NOT. 1x1 matrix

Relation of Matrix, Vector & Scalar

Matrix: $m \times n$ rectangular

Vector: $m \times 1$ or $1 \times n$ matrix

Scalar: 1×1 matrix

A scalar is a special vector

A vector is a special matrix

Matrix Conventions

- Column of a matrix $\mathbf{a}_j = \begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix}$

- Row of a matrix $\underline{\mathbf{a}^{(i)}} = \begin{bmatrix} A_{i1} & \cdots & A_{in} \end{bmatrix}$

Exercise: How to write A, in terms of the above notation?

Matrix v.s. Column Vectors

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Exercise. Write A in terms of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

Wrong: $A = \vec{a}_1 + \vec{a}_2 + \vec{a}_3$ Imperfect: $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$

Correct: $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$

matrix = row-vector of column-vectors

= "row-vector" with each entry being one column-vector

Here, we use a broader notion of "vector"

where each entry does NOT need to be a number.

Will formalize later when we talk about "block matrix".

Matrix v.s. Row Vectors

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\vec{a}^{(1)} = [1, 2, 3]$$

$$\vec{a}^{(2)} = [4, 5, 6]$$

$$A = \begin{bmatrix} \vec{a}^{(1)} \\ \vec{a}^{(2)} \end{bmatrix}$$

matrix = column-vector of row-vectors

extends notion of "vector"

Definition of Row-Column Vector Product

Issue. $\underbrace{\vec{a}^{(1)} = [1, 2]}^{\text{row vector}}$. $\vec{x} = \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\text{column vector}}$.

$\underbrace{\vec{a}^{(1)} \cdot \vec{x}}_{\text{product of row vector and column vector}}$ is NOT well-defined.

Definition $\vec{a} = (a_1 \cdots a_n)$. $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$\vec{a}\vec{x} \underset{\text{is defined as}}{:=} a_1 x_1 + \cdots + a_n x_n$$

(matrix product's special case).

BTW, this is why we do recommend $\langle \vec{a}, \vec{b} \rangle$, not $\vec{a} \cdot \vec{b}$.

since $\vec{a} \cdot \vec{b}$ and $\vec{a}\vec{b}$ represent different meanings

Extending Def to General Case

$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Def-1. $Ax = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

$Ax = \underbrace{\begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}}_{\text{collection of sum of products}}$.

Def-2 $Ax = \begin{bmatrix} [1,1] \cdot [x_1, x_2] \\ [2,4] \cdot [x_1, x_2] \end{bmatrix}$

$A = \begin{bmatrix} \vec{a}^{(1)} \\ \vdots \\ \vec{a}^{(m)} \end{bmatrix}$ (row), $\vec{x}$ (column),

$Ax = \underbrace{\begin{bmatrix} \vec{a}^{(1)}\vec{x} \\ \vdots \\ \vec{a}^{(m)}\vec{x} \end{bmatrix}}_{\text{Collection of inner products}}$.

Def-3 $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2$.

$A = [\vec{a}_1, \cdots, \vec{a}_n]$. $Ax = \underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}_{\text{linear combination of columns}}$

Definition I of Matrix-Vector Product: Row-form

Definition 6.1 (Matrix-vector product Def-1)

A is $m\times n$ matrix, $\vec{b}$ is $n\times 1$ vector,

Suppose $A = \begin{bmatrix} \vec{a}^{(1)} \\ \vdots \\ \vec{a}^{(m)} \end{bmatrix}$, $A\cdot b \triangleq \begin{bmatrix} \vec{a}^{(1)}\vec{b} \\ \vdots \\ \vec{a}^{(m)}\vec{b} \end{bmatrix} \in \mathbb{R}^{m\times 1}$

Remark: $\vec{a}^{(i)}$ is a row vector, $\vec{b}$ is a column vector, thus rigorously speaking, we cannot perform an "inner product" $\langle \vec{a}^{(i)}, \vec{b}\rangle$; instead, we shall perform the inner product of $(\vec{a}^{(i)})^T$ and $\vec{b}$, which is equal to $\vec{a}^{(i)}\vec{b}$.

$[m\times n]\cdot[n\times 1] \longrightarrow m\times 1$ vector

Matrix Representation of Linear System

Consider a General Linear System ...

System of Linear Equations

$$
\boxed{\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}}
$$

$$\Downarrow$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n}$$

Matrix Representation

$$Ax = b$$

$$b := \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Definition II of Matrix-Vector Product: Column-form

Definition 6.2 (Matrix-vector product Def-2)

A is $m \times n$ matrix, $\vec{w}$ is $n \times 1$ vector.

$A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$,

Then $A \cdot \vec{w} \overset{\triangle}{=} w_1 \cdot \vec{a}_1 + \cdots + w_n \cdot \vec{a}_n$.

Remark. Compare inner product and the above.

If $\vec{u} = [u_1, u_2 \cdots, u_n] \in \mathbb{R}^{1 \times n}$, then $\vec{u} \cdot \vec{w} = w_1 u_1 + \cdots + w_n \cdot u_n$.

v.s. $A\vec{w} = w_1 \vec{a}_1 + \cdots + w_n \vec{a}_n$.

A u is a linear combination of columns.

Claim: Def. 6.1 and Def. 6.2 are equivalent (i.e. they produce the same result).

More Examples

A u

3×5 $\quad$ 5×1

same

3×1 vector

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 5 \\ -2 & 1 & 3 & 0 & -1 \\ 0 & 7 & -1 & -2 & 4 \end{bmatrix}_{3\times 5}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ -1 \end{bmatrix}_{5\times 1}$$

$$\underline{A\,u} = \begin{bmatrix} 1*1+4*(-2)+2*0+3*5+5*(-1) \\ (-2)*1+1*(-2)+3*0+0*5+(-1)*(-1) \\ 0*1+7*(-2)+(-1)*0+(-2)*5+4*(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix} \; 3\times 1$$

Or: $$A\,u = 1 \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 5 \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix}$$

Summary: Four Forms of Linear Systems

LHS: matrix-vec product (3 form)

Scalar form

$$\begin{cases} x_1 + x_2 = 12 \\ 2x_1 + 4x_2 = 94 \end{cases}$$

Row-vector form

(Unknown vector satisfies n linear equations simultaneously)

$$\begin{cases} [1, 1] \cdot [x_1, x_2] = 12 \\ [2, 4] \cdot [x_1, x_2] = 94 \end{cases}$$

Column-vector form

(Unknown combination of columns produces vector b)

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2 = \begin{bmatrix} 12 \\ 94 \end{bmatrix}$$

$$Ax = b, \text{ where } A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, b = \begin{bmatrix} 12 \\ 94 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matrix form

(Given matrix times unknown vector produces b)

Part III Idea of Elimination

Partly from Sec. 2.2

Solve System of Linear Equations

Example (Solving a 2×2 system)

$$x - 2y = 1 \quad (1)$$
$$3x + 2y = 11 \quad (2)$$

(2) − (1) × 3, get

$$(3x - 3x) + (2y - 3 \cdot (-2y)) = 11 - 3 \cdot 1$$

$$\Rightarrow \quad 8y = 8$$

$$\Rightarrow \quad y = 1$$

From (1), get $x = 2y + 1 = 3$

So solution is $(x, y) = (3, 1)$.

More General …

Example (Solving a 3×3 system)

$$\begin{aligned} 2x + 4y - 2z &= 2 \quad ① \\ 4x + 9y - 3z &= 8 \quad ② \\ -2x - 3y + 7z &= 10 \quad ③ \end{aligned}$$

Write down major steps. Elimination 消元.

$$\left.\begin{aligned} ② - ① \times 2: &\quad \boxed{\ }\, y + \boxed{\ }\, z = \boxed{\ } \\ ③ + ①: &\quad \boxed{\ }\, y + \boxed{\ }\, z = \boxed{\ } \end{aligned}\right\} \text{ 2x2 system.}$$

Key idea: Eliminate one variable x (by adding equations) to get 2x2 system.

How to solve $n \times n$ system?

First, eliminate variable x_1,

to get $(n-1) \times (n-1)$ system with variable

$x_2, \to x_n$

Second, solve $(n-1) \times (n-1)$ system

Idea is straightforward. Gaussian Elimination.

Matrix and Linear Systems

Definition (Coefficient Matrix)

Given a linear system,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The **coefficient matrix** of the system is an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n}$$

Matrix and Linear Systems

Definition (Coefficient Matrix)

Given a linear system,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

?

The **coefficient matrix** of the system is an $m \times n$ matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n}
$$

Augmented Matrix

Definition (Augmented Matrix)

Given a linear system,
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the corresponding augmented matrix is

no x_i's

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Exercise

Consider
$$\begin{aligned} 2x_1 + 3x_2 &= 3, \\ x_1 - x_2 &= 4 \end{aligned}$$

What is the coefficient matrix?
What is the augmented matrix?

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$[A \mid b] = \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & -1 & 4 \end{array}\right].$$

Summary Today

Today, we have learned:

—Formulation of systems of linear equations

—Matrix-vector product and four forms of linear system

—Gaussian elimination to solve a linear system
(Idea, not detailed procedure)

Question: **(next time)**

Does Gaussian elimination always work?