Lecture 04

Matrix Operation I: Multiplication, Transpose and Partition

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Today's Lecture: Outline

Today … Matrix Multiplication**!**

Outline:

1. Matrix multiplication: Properties
2. Matrix multiplication and transpose
3. Block partition and matrix multiplication

Today's Lecture： Learning Goals

Today … Matrix Multiplication!

After this lecture, you should be able to

1. Apply the properties of matrix multiplication

2. Apply transpose in matrix operations

3. Conduct multiplication of partitioned matrices
—Especially: Write the expressions of valid vector-vector, matrix-vector, matrix-matrix multiplication

Part 1 Matrix Multiplication And Properties

Matrix Multiplied by a Matrix

Can a matrix be multiplied by a matrix?

More than one set of weights …

First, there're more than one ways.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 15 \\ 30 & 32 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 8 & 64 \end{bmatrix}$$

$$A \text{ "}\times\text{" } B =$$

Extra requirement: want outcome to be matrix.

(NOT vector, NOT scalar)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix},$$

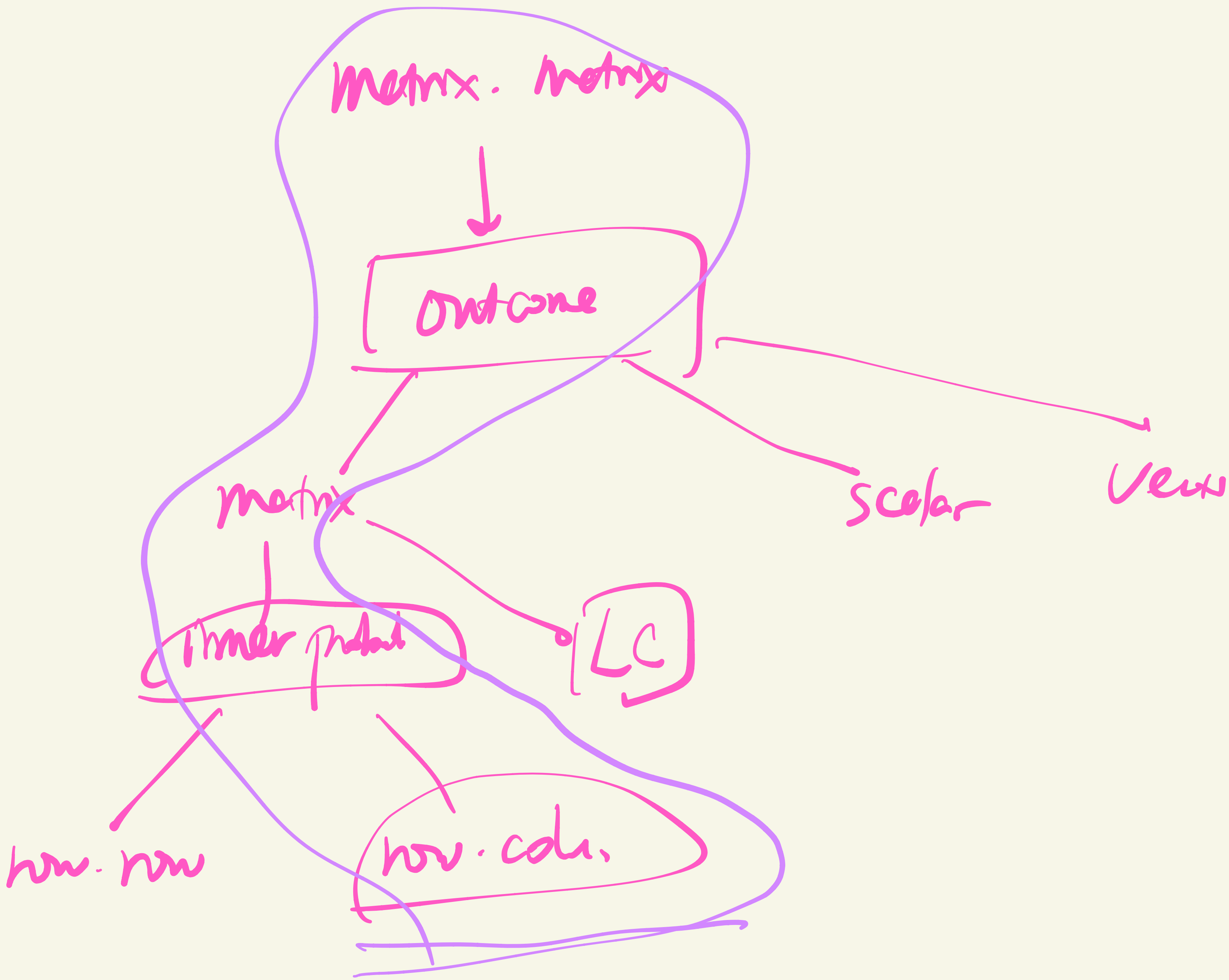
$$A \text{ "}\times\text{" } B = \begin{bmatrix} 1\times 2 & 2\times 1 \\ 3\times 8 & 4\times 7 \end{bmatrix}.$$ pointwise product.

NOT consistent with "inner product".

↓ same-size vector. same rule

Choice-1 row • row $\begin{bmatrix} (1,2)\cdot(2,1), & (1,2)\cdot(8,7) \\ (3,4)\cdot(2,1), & (3,4)\cdot(8,7) \end{bmatrix}$

✓ Choice 2. row • column $\begin{bmatrix} (1,2)\cdot(2,8), & (1,2)\cdot(1,7) \\ (3,4)\cdot(2,8), & (3,4)\cdot(1,7) \end{bmatrix}$


matrix. matrix
outcome
matrix
scalar
inner prod
LC
row . row
row . col.

Matrix Multiplied by a Matrix

Definition 4.11 (Matrix Product) Let $A \in \mathbb{R}^{m \times n}$ and $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_l] \in \mathbb{R}^{n \times r}$, then the matrix product of A by B is a $m \times r$ matrix defined by

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

Matrix Multiplied by a Matrix

Definition 4.1 (Matrix multiplication)

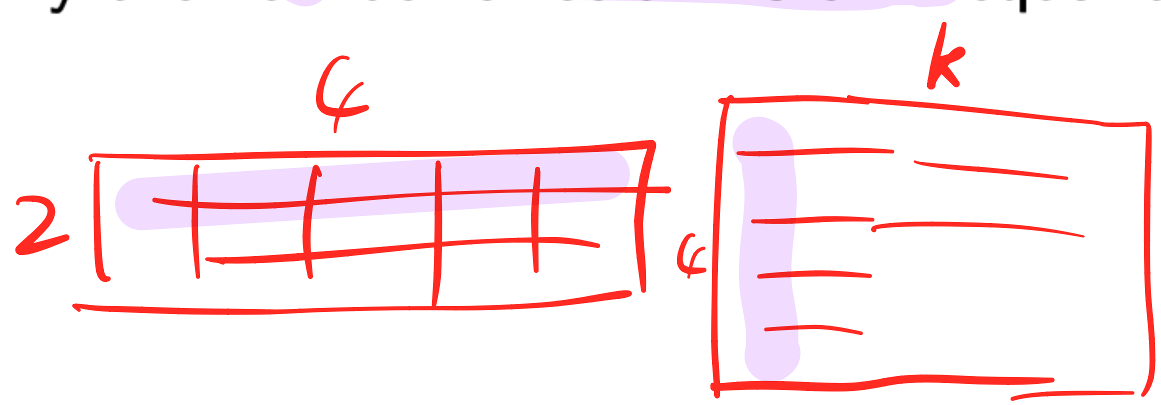
Suppose A, B are two matrices.

$$c_{ij} = \mathbf{a}^{(i)}\mathbf{b}_j = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The dimensions have to match!

Remark

1. Matrix product is a natural generalization of the matrix-vector product.
2. AB exists only and if only the number of columns of A equal to the number of rows of B.



Principle: Dimension Match

Most important principle of matrix multiplication:
Dimensions must match!

Valid multiplication: dimensions match.

Invalid multiplication: dimension do NOT match.

Exercise

A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results ?

$$BA \qquad AB \qquad ABD \qquad DC \qquad A(B+C).$$

Matrix Multiplied by a Matrix

Exercise

3×5 or 5×3 .

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix}$$

1th row, 1th col, 2, 3

(1,1) (1,2) (1,3) (1,4)

$$AB = \begin{bmatrix} 28 & 17 & 20 & 10 \\ 20 & -13 & -3 & -1 \\ -18 & -44 & 12 & -3 \end{bmatrix}$$

How Many Multiplications? (Computation Complexity)

Suppose A, B are two matrices. $A: m \times n$, $B: n \times k$

To compute AB, how many multiplications are needed?

$A = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$, $B = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$

$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = a \cdot c + b \cdot d$

$A \cdot B = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$ (2 × 4)

# of multiplications is $2 \times 4 \times 2 = 16$.

mnk

A ($m \times n$) · B ($n \times k$) = ($m \times k$)

$mk \cdot$ (# of multiplications for each)

$= mk \cdot n = mnk$.

Matrix Multiplication

$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$: identity matrix

Properties Let $A \in \mathbb{R}^{m \times n}$, $I \in \mathbb{R}^{n \times n}$ Is an identity matrix

"I" is like "1" in the number.

$AI = A$?

verify yourself

Multiplied by a zero matrix?

$\mathbf{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$

A **0** = 0

Matrix Multiplication Properties with Bugs

Properties Let $A \in \mathbb{R}^{m \times n}$, $B, C \in \mathbb{R}^{n \times l}$, $\alpha \in \mathbb{R}$

(1) $A(B + C) = AB + AC$ Left Distributive

(2) $(B + C)A = BA + CA$ Right Distributive

分配律

(3) $(AB)^T = B^T A^T$ transpose

(4) $\alpha(AB) = (\alpha A)B = A(\alpha B)$ Scalar Associative 结合律

(5) $(AB)C = A(BC)$ Associative → 结合律

$B \in \mathbb{R}^{n \times k}$

$C \in \mathbb{R}^{k \times l}$

Find Bugs?

Matrix Multiplication Properties

Properties Suppose A, B, C are matrices with proper dimensions, $\alpha \in \mathbb{R}$

(1) $A(B+C) = AB + AC$ Left Distributive

(2) $(B+C)A = BA + CA$ Right Distributive

(3) $(AB)^T = B^T A^T$

(4) $\alpha(AB) = (\alpha A)B = A(\alpha B)$ Scalar Associative

(5) $(AB)C = A(BC)$ Associative

In (1), (3), (4): $A, B \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{n \times k}$.
In (2): $A \in \mathbb{R}^{n \times k}, B, C \in \mathbb{R}^{m \times n}$.
In (5): $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times k}, B \in \mathbb{R}^{k \times p}$.

Matrix Multiplication Property 1: Proof

same as left-multiply columns

Properties Let $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times l}$, $\alpha \in \mathbb{R}$

(1) $A(B + C) = AB + AC$

$AB = A[b_1, \cdots, b_l] = [Ab_1, \cdots, Ab_l]$ → require proof

Proof of (1): (The others are exercises)

Suppose

$$B = [\mathbf{b}_1, \cdots, \mathbf{b}_l], C = [\mathbf{c}_1, \cdots, \mathbf{c}_l],$$

then

$$B + C = [(\mathbf{b}_1 + \mathbf{c}_1),\ (\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ (\mathbf{b}_l + \mathbf{c}_l)].$$

$$A(B + C) = [A(\mathbf{b}_1 + \mathbf{c}_1),\ A(\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ A(\mathbf{b}_l + \mathbf{c}_l)].$$

On the other hand,

$$AB = [A\mathbf{b}_1,\ A\mathbf{b}_2,\ \cdots\ ,\ A\mathbf{b}_l], \quad AC = [A\mathbf{c}_1,\ A\mathbf{c}_2,\ \cdots\ ,\ A\mathbf{c}_l].$$

to prove $AB + AC = A(B+C)$

only need to prove:

$$A\vec{b}_1 + A\vec{c}_1 = A(\vec{b}_1 + \vec{c}_1) = \begin{bmatrix} \vec{a}_1^T(\vec{b}_1 + \vec{c}_1) \\ \vdots \\ \vec{a}_m^T(\vec{b}_1 + \vec{c}_1) \end{bmatrix}$$

Two ways. ① use scalar definitions to check;

② split A to rows.

$$A = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}, \quad A\vec{b}_1 = \begin{bmatrix} \vec{a}_1^T b_1 \\ \vdots \\ \vec{a}_m^T b_1 \end{bmatrix}, \quad A\vec{c}_1 = \begin{bmatrix} \vec{a}_1^T \vec{c}_1 \\ \vdots \\ \vec{a}_m^T \vec{c}_1 \end{bmatrix},$$

Only need to prove: $\vec{a}_1^T b_1 + \vec{a}_1^T \vec{c}_1 = \vec{a}_1^T(\vec{b}_1 + \vec{c}_1)$

✓ inner product distributive rule

Matrix Multiplication Property 1: Proof (cont'd)

Properties Let $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times l}$, $\alpha \in \mathbb{R}$

(1) $A(B+C) = AB + AC$

Proof of (1): (The others are exercises)

Suppose

$$B = [\mathbf{b}_1, \cdots, \mathbf{b}_l], C = [\mathbf{c}_1, \cdots, \mathbf{c}_l],$$

then

$$B + C = [(\mathbf{b}_1 + \mathbf{c}_1),\ (\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ (\mathbf{b}_l + \mathbf{c}_l)].$$

Thus

$$\begin{aligned} AB + AC &= [A\mathbf{b}_1,\ A\mathbf{b}_2,\ \cdots\ ,\ A\mathbf{b}_l] + [A\mathbf{c}_1,\ A\mathbf{c}_2,\ \cdots\ ,\ A\mathbf{c}_l] \\ &= [A(\mathbf{b}_1 + \mathbf{c}_1),\ A(\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ A(\mathbf{b}_l + \mathbf{c}_l)] \\ &= A(B+C) \end{aligned}$$

Matrix Multiplication Property 3: Proof

Properties Let $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times l}$, $\alpha \in \mathbb{R}$

(3) $(AB)^T = B^T A^T$

Proof of (3): (The others are exercises)

$A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n}.$

The (i,j)-entry of AB is

and the (j,i)-entry of $(AB)^T$ is

(j,i)-entry of $B^T A^T$ is

Matrix Multiplication Property 3: Proof (cont'd)

Properties Let $A \in \mathbb{R}^{m \times n}$, $B, C \in \mathbb{R}^{n \times l}$, $\alpha \in \mathbb{R}$

(3) $(AB)^T = B^T A^T$

Proof of (3): (The others are exercises)

$A = (a_{ij})_{m \times n}, B = (b_{ij})_{m \times n}.$

The (i, j)-entry of AB is $\sum_{k=1}^{n} a_{ik} b_{kj}$,

and the (j, i)-entry of $(AB)^T$ is $\sum_{k=1}^{n} a_{ik} b_{kj}$.

(j, i)-entry of $B^T A^T$ is $\sum_{k=1}^{n} b_{kj} a_{ik} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Thus, (j, i)-entry of $(AB)^T$ and $B^T A^T$ are the same.

Missing Property of Matrix Multiplication

You might be tired of seeing these properties over and over again.

Of course, these properties should hold for any product, right?

Question:

What property does matrix multiplication NOT have?

[Compared to: scalar product; vector inner product]

Matrix Multiplication: AB and BA are NOT Equal

1. AB exists does not imply that BA exists.
Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

BA does not exist.

2. Even if both AB and BA exists, they are generally not equal ($AB \neq BA$).
Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Question: Do there exist A, B such that AB = BA?

Missing Property of Matrix Multiplication

What property does matrix multiplication NOT have?

[Compared to: scalar product; vector inner product]

Answer: matrix multiplication is NOT commutative.

In Chinese: 矩阵乘积不是可交换的。

non-commutative algebra →

非交换代数.

Rules for Vector and Matrix Multiplication

3 Rules of Real Number Multiplication: commutative, associative, distributive,.

In Chinese: 交换律,　　结合律,　　分配律

	commutative	Associative	Distributive
Real number multiplication	Yes	Yes	Yes
Vector inner product	✓	N/A	✓
Matrix multiplication	✗	✓	✓
…			

N/A means: not applicable.

$$(v\cdot w)\cdot u = v\cdot(w\cdot u)$$

scalar　vector

scalar version $\alpha(v\cdot w) = (\alpha v)\cdot w$

Part 2 Transpose and Matrix Multiplication

Matrix Transpose

Definition (Transpose)

Let $A = (a_{ij})_{m\times n}$, then the transpose of A is the matrix $B = (b_{ji})_{n\times m}$, where

$$b_{ji} = a_{ij} (i = 1, \cdots, m, j = 1, \cdots, n).$$

Notation: $B = A^T$.

Matrix Transpose

Exercise Find the transpose of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A^T = [1, 2, 3]$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & -1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 5 \\ 0 & -1 \\ 3 & 0 \end{bmatrix}$$

Transpose and Row-form, Column-form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \vdots \\ \mathbf{a}^{(m)} \end{bmatrix}$$

Exercise: write $A^\top$ using $\mathbf{a}^{(1)} \cdots \mathbf{a}^{(m)}$

incorrect

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a \cdots & & \\ \vdots & & & & \\ a_{1n} & \cdots & & & a_{mn} \end{bmatrix} = [\vec{a}^{(1)}, \cdots, \vec{a}^{(m)}]$$

~~$\vec{a}^{(1)T}$~~

Correct.

$$A^T = [(\vec{a}^{(1)})^T, \cdots, (\vec{a}^{(m)})^T]$$

Properties of Transpose

Properties

Let $A, B \in \mathbb{R}^{m \times n}$, $\alpha \in \mathbb{R}$, then

(1) $(A + B)^T = A^T + B^T$.
(2) $(\alpha A)^T = \alpha A^T$.
(3) $(A^T)^T = A$.

(4) $(AB)^T = \cancel{A^T B^T} \; B^T A^T$

A: $m \times n$, B: $n \times k$; A^T: $n \times m$, B^T: $k \times n$; B^T: $k \times n$, A^T: $n \times m$ ✓

Symmetric Matrix

Definition (Symmetric Matrix)

$m \times n$ $\quad$ $n \times m$ $\quad \nearrow$ $m = n$, has to be square

If a matrix $A \in \mathbb{R}^{m \times n}$ satisfies $A = A^T$ we call A is symmetric.

Examples: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 7 \end{bmatrix}$

Non-examples: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}$

Vector-vector products

Want to multiply u and v, two column vectors.

You are allowed to use transpose.

Which products are valid?

uv	$uv^\top$	$u^\top v$
vu	$v^\top u$	$vu^\top$

Vector-vector products

Exercise:
For the valid products, write down **the expression of the product**, using scalars.

$$u^T v = u_1 v_1 + \cdots + u_n v_n \in \mathbb{R}^{1\times 1}$$

$$\underset{n\times 1}{u}\ \underset{1\times n}{v^T} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [v_1, \cdots, v_n] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & & & \\ u_n v_1 & \cdots & \cdots & u_n v_n \end{bmatrix}$$

outer product.

Think: **Which of the valid products are equivalent?** Which are not?

Vector Outer Product

$m \times 1$ $1 \times n$
$x y^T$

Definition **(Outer Product of Two Vectors)** Let **x** be the column vectors (the length of **x** is m, and $\vec{\mathbf{y}}$ is the row vector with the length n, the product $\mathbf{x}\vec{\mathbf{y}}$ (called **outer product**) will result in a matrix.

$$\mathbf{x}\vec{\mathbf{y}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & & & \\ x_my_1 & x_ny_2 & \cdots & x_my_n \end{bmatrix}$$

Matrix Multiplication and Inner Product

Inner product is a special case of matrix multiplication.

Recall: three forms of inner product.

Form-1: $\langle \mathbf{u}, \mathbf{v} \rangle$

Form-2: $\mathbf{u}^\top \mathbf{v}$.
"Matrix multiplication".
Previously: told you to use this notation;
Now: it is just a special case of matrix product; so NOT a new notation.

$u^T v$: product of u^T and v.

Form-3: $\mathbf{u} \cdot \mathbf{v}$
NOT recommend!!

~~$A \cdot B$~~

Can we view $\mathbf{u}$, $\mathbf{v}$ as "matrix"?

Form-1: YES.
Require future knowledge.

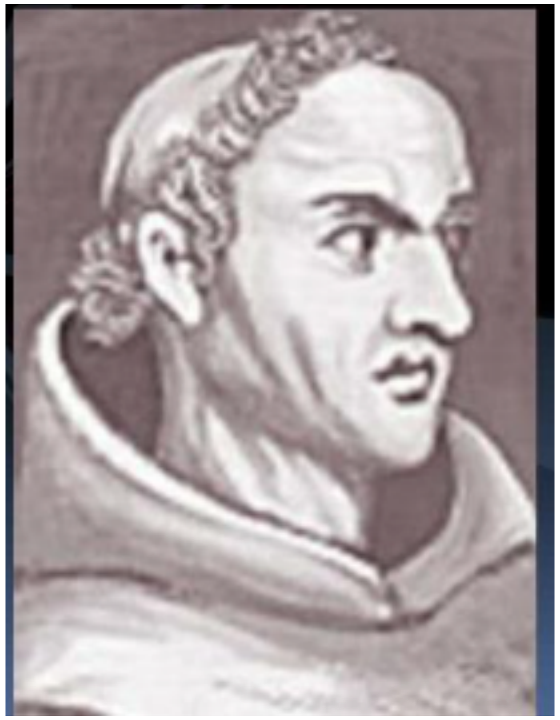
Form-2: YES.
Matrix multiplication.

Form-3: NO.
Notation only valid for vectors.

Reading: Philosophy of Notation

Occam's razor principle:

Make things as simple as possible, but not simpler.
如无必要，勿增实体
Pluralitas non est ponenda sine neccesitate.



Application 1:
Idea: Vector multiplication and matrix multiplication shall use the same notation and definition, if possible.

$\mathbf{u}^\top \mathbf{v}$ is just a matrix multiplication. So keep it.

$\mathbf{u} \cdot \mathbf{v}$ is a notation that is NOT consistent with matrix multiplication, so avoid using it.

$\langle \mathbf{u}, \mathbf{v} \rangle$ is NOT consistent with matrix multiplication, but it's consistent with "matrix inner product", so still keep it.

Then I realised (updated one)

Part 3 Block Partition and Matrix Multiplication

Matrix Partition

Sometimes, it will often be convenient to think about matrices defined in terms of other matrices.

For example, we already saw augmented matrices, defined in terms of a coefficient matrix and a vector of righthand sides.

$$[A \,|\, \mathbf{b}].$$

Another example: express a matrix by vectors

$$\begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \vdots \\ \mathbf{a}^{(m)} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

Example 1

Let's see a different example, with similar flavor of "partition".

$$P = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

What do you observe?

Can you simplify the expression?

Example 2

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

Can you simplify the expression?

4 by 6 matrix
2 by 2 blocks give
2 by 3 block matrix

Definition

The matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix}$$

is a partition of matrix with $s \times t$ blocks if the matrices A_{ij} satisfies

(1) For each fixed i, the number of rows of all A_{ij} are equal.

(2) For each fixed j, the number of columns of all A_{ij} are equal.

The matrix A_{ij} is called the (i, j)-block of A.

Make sure the blocks are the right sizes:
(i) blocks in the same (block) row have same # of rows,
(ii) blocks in the same (block) column need to have same # of columns.

Multiplication of Partitioned Matrices

Rule of thumb:

You can treat blocks as "scalars" to perform matrix-multiplication, with an extra rule:

When the blocks multiply, the dimensions must match.

Example

Block multiplication If blocks of A can multiply blocks of B, then block multiplication of AB is allowed. Cuts between columns of A match cuts between rows of B.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}. \tag{4}$$

Example 3

If

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 1 & 0 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 7 & -2 & 3 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 2 \end{bmatrix}$$

then

$$A = \left[\begin{array}{cc|ccc|c} 1 & 2 & 5 & -1 & 3 & 4 \\ 3 & 4 & -2 & 1 & 0 & 6 \\ \hline 1 & 5 & 7 & -2 & 3 & 2 \end{array}\right]$$

has the $(1,2)$-block A_{12} and $(2,3)$-block A_{23}. Moreover, the number of rows of all A_{1j} is 2, and the number of columns of all A_{i3} is 1.

Non-example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 7 & -2 & 3 \end{bmatrix}$$

Part 4 Matrix-matrix products

Strang's book Sec 2.4

Multiplication of Matrix-Vector

Matrix-vector product:

3 forms, correspond to 3 forms of linear systems.

Why 3 forms? Recall:

Matrix-vector product:

Scalar form:

Row-form:

Column-form:

Multiplication of Matrix-Vector

Matrix-vector product:

3 forms, correspond to 3 forms of linear systems.

Why 3 forms? Different ways of partitioning!

	A	x		Mimic
Valid	Scalar form m by n block	Vector N by 1	Vector m by 1	Matrix-vector product
Valid	row-form m by 1 block	vector 1 by 1 block	vector m by 1	Scalar-vector product
Valid	Column-form 1 by n block	vector n by 1 block	vector 1 by 1 block	Vector-vector product
Invalid combination	row-form m by 1 block	vector n by 1 block	ERROR!	ERROR!

Multiplication of Partitioned Matrices

What if we multiply A and B?
- Q1: What pair of partitions lead to valid multiplication?
- Q2: What results do we get?

A: whole matrix, row-partition, column-partition
B: whole matrix, row-partition, column-partition

9 combinations.

Q1: Among them, which are valid?

Multiplication of Partitioned Matrices

What if we multiply A and B?
Q1: What partitions are valid? Q2: What results do we get?

A: whole-matrix, row-partition, column-partition
B: whole-matrix, row-partition, column-partition

Q2: Expression of the valid products?

Whole-matrix Times Row-form or Column-form

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix. If B is partitioned into columns $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r]$, then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

And if A is partitioned into rows

$$\begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

then

$$AB = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{bmatrix}.$$

Important Special Case: Column-form Times Row-form

$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & b_1 & — \\ & \vdots & \\ — & b_n & — \end{bmatrix} = \begin{bmatrix} a_1 b_1 + \cdots + a_n b_n \end{bmatrix}$$

Example 7.3.4:

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Rules	Computation in Eg 7.3.4	Interpretation of product
row * column	4 dot products	**Collection of inner products**
column*row	2 outer products	**Sum of outer products**

Example: Sum of Outer Products

Example

Given

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute XY

$$XY = \left[\begin{array}{c|c} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 2 & 4 & 1 \end{array}\right] = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 10 \\ 5 & 10 & 5 \end{bmatrix}$$

Multiplication's Explanation

Form-1	Vector outer product	$[\mathbf{a}_{(1)}^\top, \ldots, \mathbf{a}_{(m)}^\top]^\top$	$[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$	$(\mathbf{a}_{(i)}^\top \mathbf{b}_j)_{m \times k}$	**Definition**
Form-2	Vector inner product	$[\mathbf{a}_1, \ldots, \mathbf{a}_n]$	$[\mathbf{b}_{(1)}^\top, \ldots, \mathbf{b}_{(n)}^\top]^\top$	$\sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_{(i)}^\top$	**Sum of outer-products**

Observation:

Form-1 mimics outer product, but consists of inner product

Form-2 mimics inner product, but consists of outer product

Multiplication of Partitioned Matrices

What if we multiply A and B?
Q1: What partitions are valid? Q2: What results do we get?

A: whole-matrix, row-partition, column-partition
B: whole-matrix, row-partition, column-partition

A: $m \times n$; B: $n \times k$.

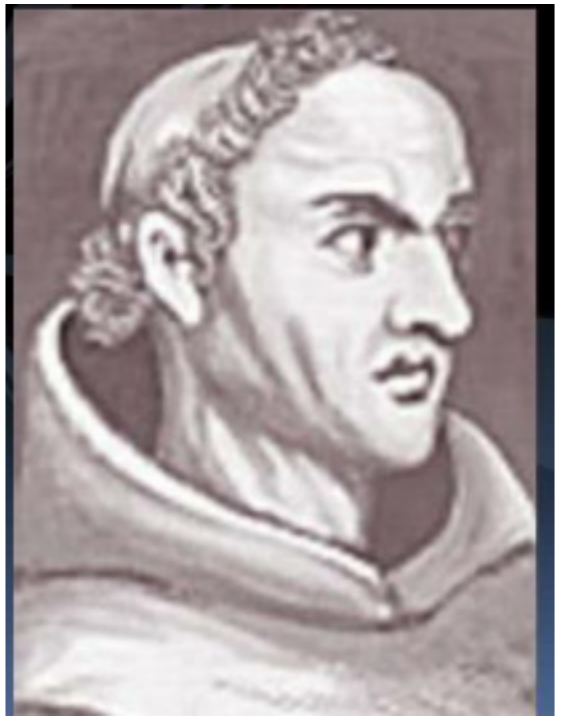
Mimic	Form of A	Form of B	Product	Remark
scalar * scalar	A	B	AB	
scalar * vector	A	$[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$	$[A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_k]$	
vector * scalarr	$[\mathbf{a}_{(1)}^\top, \ldots, \mathbf{a}_{(m)}^\top]^\top$	B	$[\mathbf{a}_{(1)}^\top B, \ldots, \mathbf{a}_{(m)}^\top B]^\top$	
Vector outer product	$[\mathbf{a}_{(1)}^\top, \ldots, \mathbf{a}_{(m)}^\top]^\top$	$[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$	$(\mathbf{a}_{(i)}^\top \mathbf{b}_j)_{m\times k}$	**Definition**
Vector inner product	$[\mathbf{a}_1, \ldots, \mathbf{a}_n]$	$[\mathbf{b}_{(1)}^\top, \ldots, \mathbf{b}_{(n)}^\top]^\top$	$\sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_{(i)}^\top$	**Sum of outer-products**

Practice Question: When is each form valid?
Where does "dim-checking" play a role?

Reading: Philosophy of Multiplication Rules

Occam's razor principle:

Make things as simple as possible, but not simpler.
如无必要，勿增实体
Pluralitas non est ponenda sine neccesitate.



Application 2:
Idea: Partitioned matrix multiplication shall use the same rules as "non-partitioned multiplication rules" if possible:

—matrix-vector-multiplication
—vector-vector-multiplication
—Scalar vector multiplication
—…

Fortunately, this is the case!!
(As we have seen in the past few pages)

Short summary of the past 2 lectures

Lec 03: matrix-vector multiplication

Represent $Ax = b$

Lec 04 Part 1-3: matrix-matrix multiplication

(+ transpose; partition)

Will be useful later for representing GE

Lec 04 part 4:

Row/column-partitioned matrix multiplication;

—matrix as sum of outer-products

Useful in SVD (~ lec 23)

Concluding Section

Summary Today

Write your summary below.

One sentence summary:

Detailed summary:

Summary Today Instructor's summary

One sentence summary:

We learned (deeper) matrix multiplication, transpose and partition.

Detailed summary:

—**Matrix multiplication and properties**:

Two rules: Row times columns; Dimension match.

Properties: Non-commutative; associative, distributive.

—**Transpose**.

Transpose and matrix operations (sum, product, partition).

Valid vector-vector products: outer product.

—**Block partition:**

Rules of partition and multiplication.

Valid partitioned matrix-vector multiplication.

Valid partitioned matrix-matrix multiplication.