Lecture 05

Matrix Operation III: Partitioned Product

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1) Reminder: 10% of score.

- Other rules may come later.

2) If the app doesn't work,
ask TA during the break.

Today's Lecture: Outline

Today … Continue Matrix Multiplication

Outline:

1. Block partition

2. Partitioned matrix multiplication

Today's Lecture: Learning Goals

Today … Matrix Multiplication!

After this lecture, you should be able to

1. Conduct multiplication of partitioned matrices

2. Especially: Write the expressions of valid matrix-matrix multiplication

using row/col forms

Part 1 Block Partition and Matrix Multiplication

Matrix Partition

Sometimes, it will often be convenient to think about matrices defined in terms of other matrices.

For example, we already saw augmented matrices, defined in terms of a coefficient matrix and a vector of righthand sides.

$$[A \mid \mathbf{b}].$$

$2x_1 + x_2 = 3$
$x_1 + 3x_2 = 4$

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 3 & 4 \end{array}\right]$$

Another example: express a matrix by vectors

row for

$$\begin{bmatrix} \mathbf{a}^{(1)} \\ \hline \mathbf{a}^{(2)} \\ \vdots \\ \mathbf{a}^{(m)} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

column form

$A = \underbrace{[a_1, a_2, \cdots a_n]}$.

"vector" of vectors

or: vector with entries being vectors.

Vector in narrow sense: array of numbers

Vector in broad sense: array of ANYTHING

$\begin{bmatrix} \text{Xu} \\ \text{Luo} \end{bmatrix}$.

Matrix in narrow sense: rectangular array of numbers.

Matrix in broad sense: rectangular array of ANYTHING → can be matrix equipped with operations

Classroom = array of students

Good definition with extra operations.

Example 1

Let's see a different example, with similar flavor of "partition".

$$P = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

What do you observe?

Can you simplify the expression?

want to save space.

Define $\vec{a} = [-1, -1, -1]$, $\vec{b} = [2, 2, 2]$

$$P = \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{a} \\ \vec{b} \end{bmatrix}$$

Define $B = \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}$

then $P = \begin{bmatrix} B \\ B \end{bmatrix}$ $\boxed{2\times 1}$ 2×3 2×3

Example 2

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

Benifit of using "block":

1) Save space;
2) Clear structure. (reduce mistake).

Can you simplify the expression?

4 by 6 matrix
2 by 2 blocks give
2 by 3 block matrix

Define $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, 2x2 identity matrix

then $A = \begin{bmatrix} I_2 & I_2 & I_2 \\ I_2 & I_2 & I_2 \end{bmatrix}$

Define $J = \begin{bmatrix} I_2 \\ I_2 \end{bmatrix}$, then $A = [J \; J \; J]$

$$\hat{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

same as last page.?

<u>No</u>

Definition

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

The matrix

"block-row"

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix}$$

or partitioned matrix

is a partition of matrix with $s \times t$ blocks if the matrices A_{ij} satisfies

(1) For each fixed i, the number of rows of all A_{ij} are equal.

(2) For each fixed j, the number of columns of all A_{ij} are equal.

The matrix A_{ij} is called the (i, j)-block of A.

Make sure the blocks are the right sizes:
(i) blocks in the same (block) row have same # of rows,
(ii) blocks in the same (block) column need to have same # of columns.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} I_2 & I_2 & I_2 \\ I_3 & I_3 & I_3 \end{bmatrix}.$$

Is this valid? <u>No</u>: I_2: 2 col's, I_3: 3 col's.

What about

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & & & 1 & & \\ & 1 & & & 1 & \\ & & 1 & & & 1 \end{array}\right] = \underbrace{\begin{bmatrix} I_2 & I_2 & I_2 \\ I_3 & I_3 & \end{bmatrix}}_{\text{not "rectangular".}}$$

Pyramid

not a matrix

Multiplication of Partitioned Matrices

Rule 1: block-form dim match, e.g. $\boxed{2\times 2} \cdot \boxed{2\times 1}$ match in (4).

Rule of thumb:

You can treat blocks as "scalars" to perform matrix-multiplication, with an extra rule:

When the blocks multiply, the dimensions must match.

Rule 2. component dim match.

e.g. $A_{11}B_{11}$ should be valid

Example

Block multiplication If blocks of A can multiply blocks of B, then block multiplication of AB is allowed. Cuts between columns of A match cuts between rows of B.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}. \tag{4}$$

component dim is the dimension of A_{ij}

$\boxed{2\times 2}$ $\boxed{2\times 1}$

block-form dim of A is 2x2.

Relation of Different component products

Assumption. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is partitioned matrix; $B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ is partitioned matrix.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

Requirements.

(a) $A_{11}B_1$ is valid. $\Longleftrightarrow$ $\#\,\mathrm{col}(A_{11}) = \#\,\mathrm{row}(B_1)$

(b) $A_{12}B_2$ is valid $\Longleftrightarrow$ $\#\,\mathrm{col}(A_{12}) = \#\,\mathrm{row}(B_2)$.

(c) $A_{21}B_1$ is valid.

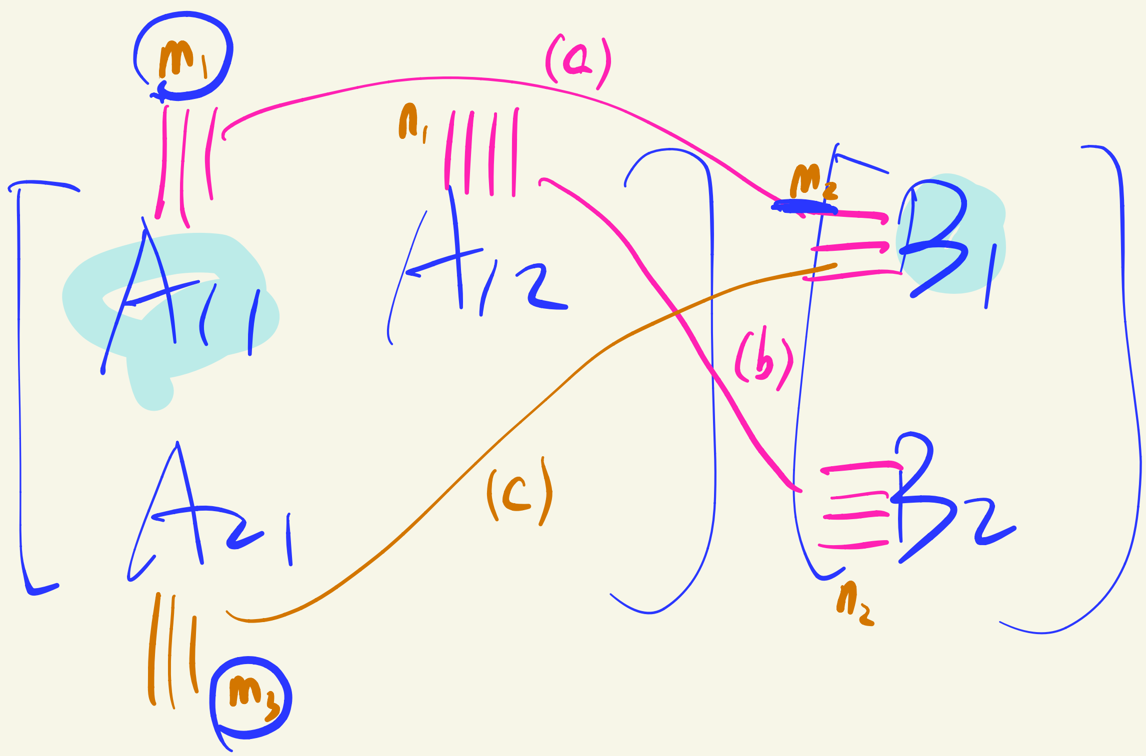
(d) $A_{21}B_2$ is valid.

Exercise.

Q1. Does (a) $\Rightarrow$ (b) ? 85% no. NO

Q2 Does (a) $\Rightarrow$ (c) ? YES.

(a) $\Longleftrightarrow$ (c), (b) $\Longleftrightarrow$ (d).



(a) $\Longleftrightarrow$ $m_1 = m_2$; (b) $\Longleftrightarrow$ $n_1 = n_2$; (c) $\Longleftrightarrow$ $m_1 = m_3$.

(c) & (b) are not directly related.

(a) $\Longleftrightarrow$ (c) : because $m_1 = m_3$.

Example 3

If

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 1 & 0 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 7 & -2 & 3 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 2 \end{bmatrix}$$

then

$$A = \left[\begin{array}{cc|ccc|c} 1 & 2 & 5 & -1 & 3 & 4 \\ 3 & 4 & -2 & 1 & 0 & 6 \\ \hline 1 & 5 & 7 & -2 & 3 & 2 \end{array}\right]$$

block row has 2 rows

$\boxed{2 \times 3}$

block row has 1 row

has the $(1,2)$-block A_{12} and $(2,3)$-block A_{23}. Moreover, the number of rows of all A_{1j} is 2, and the number of columns of all A_{i3} is 1.

Non-example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 7 & -2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 5 \end{bmatrix} \end{bmatrix} \quad 2 \times 1$$

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{bmatrix} \quad 1 \times 1$$

2×2

Part 2 Matrix-matrix products

Strang’s book Sec 2.4

Multiplication of Matrix-Vector

Matrix-vector product:

3 forms, correspond to 3 forms of linear systems.

Why 3 forms? Recall:

Matrix-vector product: $A\vec{x}$, A: 2×2, x: 2×1.

Scalar form: $A\vec{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$

Row-form: $A\vec{x} = \begin{bmatrix} \vec{a}^{(1)} \\ \vec{a}^{(2)} \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}^{(1)}\vec{x} \\ \vec{a}^{(2)}\vec{x} \end{bmatrix}$

where $\vec{A} = \begin{bmatrix} \vec{a}^{(1)} \\ \vec{a}^{(2)} \end{bmatrix}$

Column-form:

$A\vec{x} = [\vec{a}_1, \vec{a}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2$.

where $A = [\vec{a}_1, \vec{a}_2]$

Comments

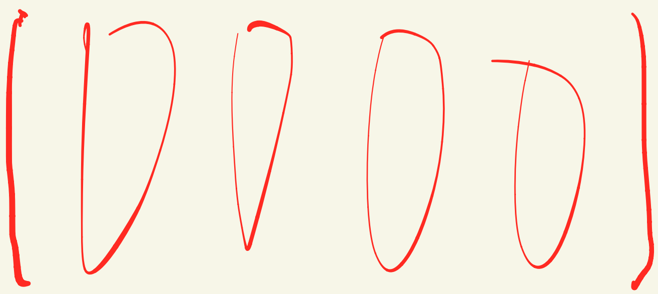
① All forms of matrix-vector products are special cases of partioned matrix products.

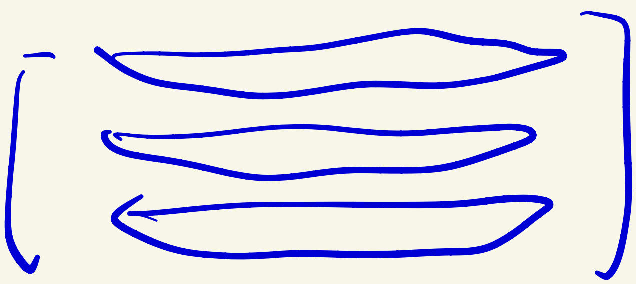
② Different partitions of A and x lead to different forms

③ There can be 10 different ways, these 3 are the simplest.

Different ways of partioning A:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

$A =$ 

$A =$ 

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right]$$

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right]$$

Row Form of Ax: Interpretation

We will demonstrate: row form of Ax is just a special case of portioned matrix product.

Row-form:

special case of partitioned matrix product

$$A = \begin{bmatrix} \vec{a}^{(1)} \\ \vec{a}^{(2)} \end{bmatrix}, \quad \vec{x}, \quad A\vec{x} = \begin{bmatrix} \vec{a}^{(1)}\vec{x} \\ \vec{a}^{(2)}\vec{x} \end{bmatrix}$$

block-form dim: $\boxed{2\times 1}$ $\boxed{1\times 1}$

similar form
形式相同

Scalar-form analogy: $\hat{A} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, x \in \mathbb{R}, \hat{A}\cdot\hat{x} = \begin{bmatrix} \alpha_1\hat{x} \\ \alpha_2\hat{x} \end{bmatrix}$

Column Form of $A x$: Interpretation

$A = [\vec{a}_1, \vec{a}_2]$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Step 1:

block-form dim $\boxed{1 \times 2}$ $\boxed{2 \times 1}$

If you forget $A\vec{x}$ in this form.

Step 2:

Scalar form analogy: $\hat{A} = \underbrace{[\alpha_1, \alpha_2]}_{1\times 2}$, $\hat{x} = \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{2\times 1}$, $\hat{A}\hat{x} = \underline{\alpha_1 \beta_1 + \alpha_2 \beta_2}$.

LC of α_1 and α_2 (scalars)

Step 3: $A\vec{x} = \underbrace{[\vec{a}_1, \vec{a}_2]}_{\text{vec}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{vec}} = \vec{a}_1 x_1 + \vec{a}_2 x_2$.

LC of columns of A.

Multiplication of Matrix-Vector

Unified View, via matrix partition

Matrix-vector product:

3 forms, correspond to 3 forms of linear systems.

Why 3 forms? Different ways of partitioning!

teach you: (1) matrix partitions (2) product rules

↓ lead to 3 form.

	A	x		Mimic
Valid	Scalar form m by n block	Vector N by 1	Vector m by 1	Matrix-vector product
Valid	row-form m by 1 block	vector 1 by 1 block	vector m by 1	Scalar-vector product
Valid	Column-form 1 by n block	vector n by 1 block	vector 1 by 1 block	Vector-vector product
Invalid combination	row-form m by 1 block	vector n by 1 block	ERROR!	ERROR!

$\begin{bmatrix} a^{(1)}x \\ a^{(2)}x \end{bmatrix}$

$\sum a_i \cdot x_i$

$A = \begin{bmatrix} a^{(1)} \\ a^{(2)} \end{bmatrix}$

What's the message in the previous page?

1) All forms of matrix-vector products are special cases of portioned matrix products.
[mentioned before; repeat to emphasize]

2) Had you learned (a) partition;
(b) product of partition,
you would be able to derive 3 forms of matrix-vector products yourself!

Multiplication of Partitioned Matrices

$A: 2\times 3,\quad B: 3\times 2$

AB is valid.

What if we multiply A and B?
    Q1: What pair of partitions lead to valid multiplication?
    Q2: What results do we get?

A $\qquad \begin{bmatrix} a_{(1)}^T \\ a_{(2)}^T \end{bmatrix} \qquad [a_1, a_2, a_3]$

A: whole matrix, row-partition, column-partition

B: whole matrix, row-partition, column-partition

B $\qquad \begin{bmatrix} b_{(1)}^T \\ b_{(2)}^T \\ b_{(3)}^T \end{bmatrix} \qquad [b_1, b_2].$

9 combinations.

Q1: Among them, which are valid?

check AB

Standard way.

	$\overset{3\times 2}{\boxed{1\times 1}}$ B	$\boxed{3\times 1}$ $\begin{bmatrix} b_{(1)}^T \\ b_{(2)}^T \\ b_{(3)}^T \end{bmatrix}$	$\boxed{1\times 2}$ $[b_1, b_2]$ (3×1, 3×1)
$\overset{2\times 3}{\boxed{1\times 1}}$ A	✓	✗	✓ → (F2) $[Ab_1, Ab_2]$
$\boxed{2\times 1}$ $\begin{bmatrix} a_{(1)}^T \\ a_{(2)}^T \end{bmatrix}$	✓ (F3)	✗	✓ (F1) → $\begin{bmatrix} a_{(1)}^T b_1 & a_{(1)}^T b_2 \\ a_{(2)}^T b_1 & a_{(2)}^T b_2 \end{bmatrix}$ Definition
$\boxed{1\times 3}$ $[a_1, a_2, a_3]$	✗	✓ (F4)	✗

Form-2: (Matrix) · (Column-Form)

Q: Is $A \cdot [b_1, b_2]$ valid?

$\boxed{1\times1}$ $\boxed{1\times2}$

Step 1 Write block-form dim [rule 1]

1=1, so match.

Step 2 Perform product.

If valid, $A[b_1, b_2] \overset{\text{should}}{=\!=} [Ab_1, Ab_2]$.

Step 3. Check component dim [rule 2]

i.e. check Ab_1, Ab_2 are valid or not

2×3 3×1

Form 3: (Row-form) · Matrix

Step 2 $\begin{bmatrix} \underbrace{a_{(1)}^T}_{} \\ a_{(2)}^T \end{bmatrix} \cdot B$

$a_{(1)}^T \to 1\times 3$, $a_{(2)}^T \to 1\times 3$, $B \to 3\times 2$

should

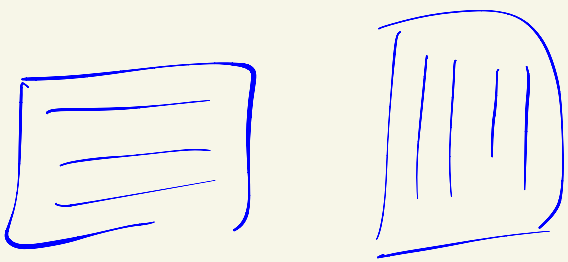
$$\begin{array}{cc} 1\times 3 & 3\times 2 \\ \end{array}$$
$$\begin{bmatrix} a_{(1)}^T B \\ a_{(2)}^T B \end{bmatrix}$$

Step 3. $3=3$, so match.

Remark: Many students (>30%) thought this is invalid.

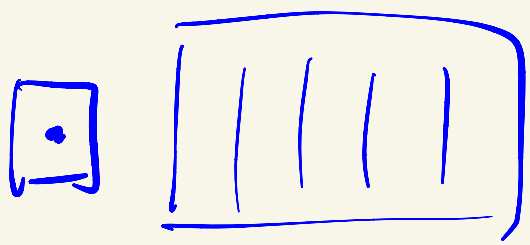
Visualize Three Valid Forms

Form 1. Def.



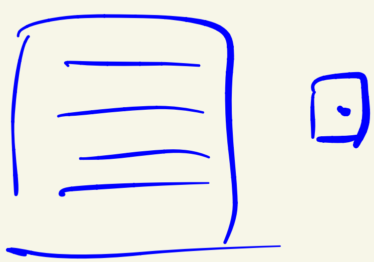
(rows) × (columns)

Form 2.



(matrix) × (columns)

Form 3.



(rows) × (matrix)

Form 4.

columns × rows.

UV^T

$$[a_1, a_2, a_3]\begin{bmatrix} b_{(1)}^T \\ b_{(2)}^T \\ b_{(3)}^T \end{bmatrix}$$

$$= a_1 b_{(1)}^T + a_2 b_{(2)}^T + a_3 b_{(3)}^T$$

sum of outer products..

Last time Valid matrix products of vector–vector

$u^T v$	row times column	✓	F1
$u v^T$	column times row	✓	F4
$u v$	column times column	✗	
$u^T v^T$	row times row	✗	

Tricky Case: "1×1" in Block Form Dimension

$A = [a_1, a_2, a_3]$, B (column-form)·(matrix)

$\boxed{1\times 3}$ $\quad$ $\boxed{1\times 1}$

$2\times 1 \quad 2\times 1 \quad 2\times 1 \qquad 3\times 2$.

$3 \neq 1$, so seems not valid.

Step 2 But: scalar-vector product (NOT matrix product)

$$[\underset{1\times 1}{\alpha_1}, \underset{1\times 1}{\alpha_2}, \underset{1\times 1}{\alpha_3}] \cdot \underset{1\times 1}{b} = [\alpha_1 b, \alpha_2 b, \alpha_3 b]$$

But cannot apply.

Step 3 If valid, $AB = [\underset{2\times 1}{a_1} \underset{3\times 2}{B}, a_2 B, a_3 B]$

does NOT match

Multiplication of Partitioned Matrices

What if we multiply A and B?
Q1: What partitions are valid? Q2: What results do we get?

A: whole-matrix, row-partition, column-partition
B: whole-matrix, row-partition, column-partition

Q2: Expression of the valid products?

Whole-matrix Times Row-form or Column-form

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix. If B is partitioned into columns $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r]$, then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

And if A is partitioned into rows

$$\begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

then

$$AB = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{bmatrix}.$$

Important Special Case: Column-form Times Row-form

$$\begin{bmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \boldsymbol{b}_1 & - \\ & \vdots & \\ - & \boldsymbol{b}_n & - \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}$$

Example 7.3.4:

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Rules	Computation in Eg 7.3.4	Interpretation of product
row * column	4 dot products	**Collection of inner products**
column*row	2 outer products	**Sum of outer products**

Example: Sum of Outer Products

Example

Given

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute XY

$$XY = \left[\begin{array}{c|c} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 2 & 4 & 1 \end{array}\right] = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 10 \\ 5 & 10 & 5 \end{bmatrix}$$

Multiplication's Explanation

Form-1	Vector outer product	$[\mathbf{a}_{(1)}^\top, \ldots, \mathbf{a}_{(m)}^\top]^\top$	$[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$	$(\mathbf{a}_{(i)}^\top \mathbf{b}_j)_{m \times k}$	**Definition**
Form-2	Vector inner product	$[\mathbf{a}_1, \ldots, \mathbf{a}_n]$	$[\mathbf{b}_{(1)}^\top, \ldots, \mathbf{b}_{(n)}^\top]^\top$	$\sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_{(i)}^\top$	**Sum of outer-products**

Observation:

Form-1 mimics outer product, but consists of inner product

Form-2 mimics inner product, but consists of outer product

Multiplication of Partitioned Matrices

What if we multiply A and B?
Q1: What partitions are valid? Q2: What results do we get?

A: whole-matrix, row-partition, column-partition
B: whole-matrix, row-partition, column-partition

A: $m \times n$; B: $n \times k$.

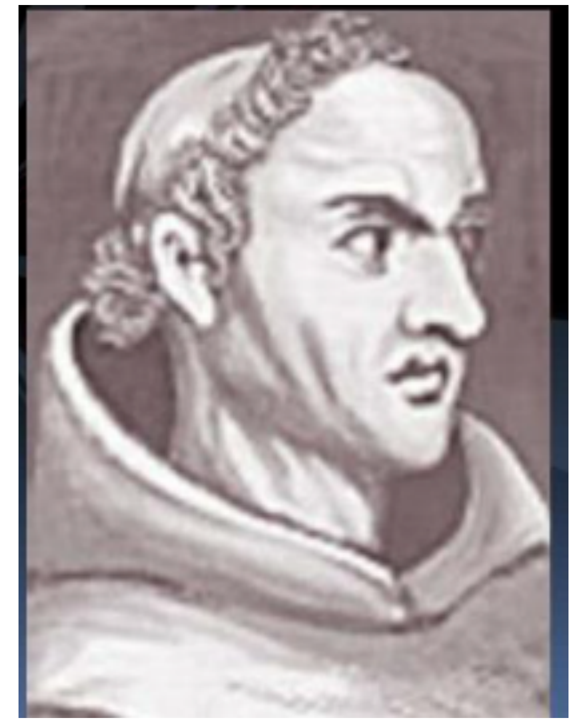
	Mimic	Form of A	Form of B	Product	Remark
	scalar * scalar	A	B	AB	
F2	scalar * vector	A	$[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$	$[A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_k]$	
F3	vector * scalarr	$[\mathbf{a}_{(1)}^\top, \ldots, \mathbf{a}_{(m)}^\top]^\top$	B	$[\mathbf{a}_{(1)}^\top B, \ldots, \mathbf{a}_{(m)}^\top B]^\top$	
F1	Vector outer product	$[\mathbf{a}_{(1)}^\top, \ldots, \mathbf{a}_{(m)}^\top]^\top$	$[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k]$	$(\mathbf{a}_{(i)}^\top \mathbf{b}_j)_{m\times k}$	**Definition**
F4	Vector inner product	$[\mathbf{a}_1, \ldots, \mathbf{a}_n]$	$[\mathbf{b}_{(1)}^\top, \ldots, \mathbf{b}_{(n)}^\top]^\top$	$\sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_{(i)}^\top$	**Sum of outer-products**

Practice Question: When is each form valid?
Where does "dim-checking" play a role?

Reading: Philosophy of Multiplication Rules

Occam's razor principle:

Make things as simple as possible, but not simpler.
如无必要，勿增实体
Pluralitas non est ponenda sine neccesitate.



Application 2:

Idea: Partitioned matrix multiplication shall use the same rules as "non-partitioned multiplication rules" if possible:

—matrix-vector-multiplication
—vector-vector-multiplication
—Scalar vector multiplication
—…

dim matching

Fortunately, this is the case!!
(As we have seen in the past few pages)

Short summary of the past 2 lectures

Lec 03: matrix-vector multiplication

Represent Ax = b

Lec 04: matrix-matrix multiplication

(+ transpose)

Will be useful later for representing GE

Lec 05

Row/column-partitioned matrix multiplication;

—matrix as sum of outer-products

Useful in SVD (~ lec 23)

Concluding Section

Summary Today

Write your summary below.

One sentence summary:

Detailed summary:

Summary Today Instructor's summary

One sentence summary:

We learned (deeper) matrix multiplication and partition of matrix.

Detailed summary:

—**Block partition:**

Rules of partition and multiplication.

Valid partitioned matrix-matrix multiplication.

(rows) · (columns) : Definition

(matrix) · (columns)

(rows) · (matrix)

(columns) · (rows) = sum of outer product