Lecture 06

Solving Linear System: $n \times n$ *System*

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Recall

Lecture 3

In ~~the last lecture~~ … (How to solve *simple* linear systems)

- Definitions of linear equations and systems of linear equations

- Examples of solving 2×2 system of linear equations

- Definition of an augmented matrix representation

Today's Lecture

Today … More on System of **Linear Equations!**

After this lecture, you should be able to

1. Tell what are elementary row operations, and why they are allowable

2. Tell the definition of lower and upper triangular matrices

3. Solve a $n \times n$ linear system (square system) using Gaussian Elimination

Part I Gaussian Elimination and Row Operations

Partly from Sec. 2.2

Recall: Augmented Matrix

Definition (Augmented Matrix)

Given a linear system,
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the corresponding augmented matrix is

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Overdetermined, Underdetermined and Square

Definition (System of Linear Equations)

An $m \times n$ system of linear equations is

2 va

5 eq

(1) **overdetermined system** if $m > n$

5 va

(2) **underdetermined** system if $m < n$

2 eq

(3) **square system** if $m = n$

3x3

2x2

5x5

Today we solve an $n \times n$ system, i.e.,

square system

An $m \times n$ matrix is

(1) tall, if $m > n$;

(2) wide, if $m < n$;

(3) square, if $m = n$.

Special Matrices

Definition (Lower Triangular Matrix) 下三角矩阵

A square matrix of the form

$$L = \begin{bmatrix} \ell_{1,1} & & & & \mathbf{0} \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \dots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

对角线
diagonal

entries below diagonal

is called a **lower triangular matrix.**

$$L_{ij} = 0 \quad \text{for all} \quad i < j$$

L_{ij} is any number, $\forall\ i \geq j$

Special Matrices

Definition (Upper Triangular Matrix) 上三角

A square matrix of the form

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

is called an **upper triangular matrix**

$$U_{ij} = 0 \quad \text{for all} \quad i > j$$

Diagonal Matrix 对角矩阵

$$D = \begin{bmatrix} \times & & & 0 \\ & \times & & \\ & & \ddots & \\ 0 & & & \times \end{bmatrix}$$

$$D_{ij} = 0, \ \forall i \neq j.$$

Special Matrices

Definition (~~positive~~ nonzero-diagonal triangular Matrix)

A matrix is called a ~~positive~~ nonzero-diagonal **triangular** matrix if

$$U_{ij} = 0 \quad \text{for all} \quad i > j \quad \text{and} \quad U_{ii} \neq 0 \quad \text{for all} \quad i \qquad \text{(upper)}$$

$$L_{ij} = 0 \quad \text{for all} \quad i < j \quad \text{and} \quad L_{ii} \neq 0 \quad \text{for all} \quad i \qquad \text{(lower)}$$

Examples

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} : \quad \text{lower triangular}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

diagonal matrix
upper
lower.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

none of these.
(Non-square)

Example of 2x2 system: Matrix Perspective

$$\begin{cases} x_1 + x_2 = 12 & ① \\ 2x_1 + 4x_2 = 38 & ② \end{cases}$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right]$$

Write augmented matrix for rest.

② − 2 × ①, get

$$\begin{cases} x_1 + x_2 = 12 & ① \\ 2x_2 = 14 & ③ \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right]$$

What do you observe?

1) upper triangular

2) diagonal.

③/2, get

$$\begin{cases} x_1 + x_2 = 12 & ① \\ x_2 = 7 & ④ \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 7 \end{array}\right]$$

Finally, ① − ④, get

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \end{array}\right]$$

order $[x_1, x_2, x_3, \ldots]$.

Observation Elimination is getting upper triangular & diagonal.

Last page. LHS: equation operation. RHS: matrices.

Can I translate Eq. operation to matrix operation?

$$\begin{cases} x_1 + x_2 = 12 & ① \\ 2x_1 + 4x_2 = 38 & ② \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right]$$

② − 2 × ①, get

Write augmented matrix for rest.

$$\begin{cases} x_1 + x_2 = 12 & ① \\ 2x_2 = 14 & ③ \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right]$$

$(2x_1 + 4x_2 = 38)$

$-\,2\,(x_1 + x_2 = 12)$

Throw away (x_1, x_2) only numbers (adding numbers)

$[2, 4 \mid 38]$

$-\,2\,[1, 1 \mid 12]$

$2x_1 - 2x_1 = 0$ $\quad 4x_2 - 2x_2 = 2x_2$ $\quad 38 - 2\cdot 12 = 14$

$2 - 2\cdot 1 = 0$ $\quad 4 - 2\cdot 1 = 2$ $\quad 38 - 2\cdot 12 = 14$

Last page. LHS: equation operation. RHS: matrices.

Can I translate Eq. operation to matrix operation?

Ignore equations.

Think: What happens to matrices?

Answer: Substract 2·(Row 1) from (Row 2)

$$\begin{array}{l} \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right] \end{array} \begin{array}{l} \to R_1 \quad \text{Row 1} \\ \to R_2 \quad \text{Row 2} \end{array}$$

$(R_2 \to) R_2 - R_1 \cdot 2$

Write augmented matrix for rest

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right]$$

Calculation:

$$\begin{aligned} & [2, 4, 38] \\ - & 2 \cdot [1, 1, 12] \\ = & [0, 2, 14] \end{aligned}$$

$$\begin{array}{rl} R_2 & [2, 4 \mid 38] \\ -2R_1 & -2[1, 1 \mid 12] \end{array}$$

$$\begin{array}{ccc} 2-2\cdot 1 & 4-2\cdot 1 & 38-2\cdot 12 \\ \| & \| & \| \\ [\,0 & 2 & 14\,] \end{array}$$

Another row operation.

$$\begin{cases} x_1 + x_2 = 12 & ① \\ 2x_2 = 14 & ③ \end{cases} \longleftrightarrow \begin{array}{c} \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \end{array}$$

$R_2/2$

③/2, get

$$\begin{cases} x_1 + x_2 = 12 & ① \\ x_2 = 7 & ④ \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 7 \end{array}\right]$$

Multiply 2nd row by $\frac{1}{2}$.

3rd Operation

$$\begin{cases} 2x_1 + 4x_2 = 38 & ① \\ x_1 + x_2 = 12 & ② \end{cases} \qquad \left[\begin{array}{cc|c} 2 & 4 & 38 \\ 1 & 1 & 12 \end{array}\right]$$

Either $② - ① \cdot \frac{1}{2}$

or $① - ② \cdot 2$, ✓ (not standard)

$$\downarrow \text{swap row } R_1 \leftrightarrow R_2$$

Swap it first: (so that keep 1st; work on 2nd)

$$\begin{cases} x_1 + x_2 = 12 & ① \\ 2x_1 + 4x_2 = 38, & ② \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right]$$

$② - ① \cdot 2$,

Review

Solving a 2×2 system:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ \boxed{4} & 5 & 14 \end{array}\right] \xRightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right] \xRightarrow{R_2 \cdot (-\frac{1}{3})} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \xRightarrow{R_1 - 2 \cdot R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$$

eliminate "4"   upper triangle   make diagonal all 1

What are the key steps?

(1) Multiply a row by a **non-zero** scalar

(2) Add to one row a scalar multiple of another

(3) Swap the positions of two rows [not used in this example]

Allowable Operations on Equations

$\alpha\ (x_1 + 2x_2 = 4)$

(1) [**Multiplication**] Multiply an equation by a **non-zero** scalar

$\alpha(x_1 + 2x_2 = 4) + (3x_1 + 5x_2 = 7)$

(2) [**Addition**] Add to one equation a scalar multiple of another

(3) [**Interchange**] Swap two equations

$x_1 + 2x_2 = 4$

$3x_1 + 5x_2 = 7$

Operations on linear equations!

Other Operations

Exercise (Other Operations)

Can the following operations be performed?

(4) Multiply a row by zero   No.

(5) Exchange variables   No. (unless extra steps are taken; skip details for now)

(6) Multiply the coefficients of two equations   No.

Allowable Operations on Matrix

$$\alpha \left[\begin{array}{cc|c} 1 & 2 & 14 \end{array}\right]$$

(1) [**Multiplication**] Multiply a row by a **non-zero** scalar

(2) [**Addition**] Add to one row a scalar multiple of another

$$\alpha \left[\begin{array}{cc|c} 1, & 2 & 14 \end{array}\right] + \left[\begin{array}{cc|c} 2 & 2 & 36 \end{array}\right]$$

(3) [**Interchange**] Swap the positions of two rows

$$\begin{array}{c} \left[\begin{array}{cc|c} 1, & 2 & 4 \end{array}\right] \\ \left[\begin{array}{cc|c} 2 & 2 & 36 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{cc|c} 2 & 2 & 36 \\ 1 & 2 & 4 \end{array}\right]$$

Operations on rows of an augmented matrix!

Allowable Operations on Rows

Definition (Elementary Row Operations) (初等行变换)

αv_i scalar-vector product

(1) [**Multiplication**] Multiply a row by a **non-zero** scalar

$\alpha v_i + v_j$ LC of vectors

(2) [**Addition**] Add to one row a scalar multiple of another

(3) [**Interchange**] Swap the positions of two rows

$[1, 2, 3) \to [\sin(1), e^2, 3^{1000}]$

is a row operation, but not elementary row operation

Allowable Operations

(1) [**Multiplication**] Multiply a row by a **non-zero** scalar

(2) [**Addition**] Add to one row a scalar multiple of another

(3) [**Interchange**] Swap the positions of two rows

Exercise (The operations preserve solutions)

Show that performing the operations above and create a new system will not vary the solution of the original system!

Claim

Suppose $\begin{cases} A_1 x = b_1 \\ A_2 x = b_2 , \end{cases}$ (P1) where $A_1, A_2 \in R^{1\times 2}$. $x \in R^{2\times 1}$

After row operation, get

$$\begin{cases} A_1 x = b_1 \\ (\alpha A_1 + A_2) x = (\alpha b_1 + b_2) . \end{cases} \quad (P2)$$

Then (P1) and (P2) has the same solution.

Proof. (C1) If x^* satisfies (P1), then x^* satisfies P2

If x^* satisfies $\begin{cases} A_1 x^* = b_1 & ① \\ A_2 x^* = b_2 & ② \end{cases}$

Then $\alpha A_1 x^* = \alpha b_1$, add to ② — property: $p=q \Rightarrow \alpha p = \alpha q$, $\forall \alpha, p, q \in R$. (M1)

$\alpha A_1 x^* + A_2 x^* = \alpha b_1 + b_2$

scalar distributive law (M2): $\alpha \vec{x} + \beta \vec{x} = (\alpha+\beta) \vec{x}$

$\Leftrightarrow (\alpha A_1 + A_2) x^* = \alpha b_1 + b_2$, so (P2) holds. (C1) done.

Need (C2): If $\hat{x}$ satisfies (P2), "similarly" (as exercise), $\hat{x}$ satisfies P1. □

Remark: Two properties are used in the proof: (M1) and (M2).

Judgement True or False.

Suppose $\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 = b_1 & ① \\ \beta_1 x_1 + \beta_2 x_2 = b_2 & ② \end{cases}$

(order-1) · order-1 = order-2

+ = order 1

After row operation,

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 = b_1 \\ (\alpha_1\beta_1) x_1 + (\alpha_2\beta_2) x_2 = b_1 b_2 . \end{cases} \quad (P2)$$

Then (P1) and (P2) has the same solution.

False.

Why does the same argument in the last page NOT work?

$\begin{cases} p = q \\ w = v \end{cases} \Rightarrow pw = qv.$

Step 1 ① · ②: $(\alpha_1 x_1 + \alpha_2 x_2)(\beta_1 x_1 + \beta_2 x_2) = b_1 b_2$.

$\not\Leftrightarrow$

order 2 $\alpha_1\beta_1 x_1^2$

order 1 linear $(\alpha_1\beta_1) x_1 + (\alpha_2\beta_2) x_2 = b_1 b_2$.

In conclusion, the 1st step holds, but the 2nd step does NOT hold.

Gaussian Elimination

top to down

Step 1: Forward Elimination (Equation)

$$x + y + z = 6 \quad ①$$
$$x + 2y + 2z = 9 \quad ②$$
$$x + 2y + 3z = 10 \quad ③$$

$$\downarrow ② - ①,\ ③ - ①$$

$$\begin{cases} x + y + z = 6 \\ y + z = 3 \quad ④ \\ y + 2z = 4 \quad ⑤ \end{cases}$$

$$\downarrow ⑤ - ④$$

$$x + y + z = 6$$
$$y + z = 3$$
$$\boxed{z = 1}$$

Plug in

Step 1: Forward Elimination (Matrix)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 9 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

$$\downarrow R_2 \leftarrow R_2 - R_1,\ R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$\downarrow R_3 \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Gaussian Elimination

down to top

Step 2: Backward Substitution (Scalar)

$$x + y + \cancel{z} = 6 \quad ①$$
$$y + \cancel{z} = 3 \quad ②$$
$$z = 1 \quad ③$$

$\downarrow$ ② − ③, ① − ③

$$\begin{cases} x + y = 5 & ④ \\ y = 2 & ⑤ \\ z = 1 \end{cases}$$

$\downarrow$ ④ − ⑤

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Step 2: ~~Backward~~ Substitution (Matrix)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\downarrow$ $R_2 - R_3$, $R_1 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\downarrow$ $R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Pipeline: Gaussian Elimination for "Good" Systems

Pipeline

towards upper triangular matrix.

α

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
\boxed{a_{21}}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
$$

$R_2 + \alpha \cdot R_1$, (to eliminate $a_{21}x_1$).

what is α?

$(\alpha a_{11}x_1 + a_{21}x_1) = 0$

$\Rightarrow \alpha = -\frac{a_{21}}{a_{11}}$

require $a_{11} \neq 0$.

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
0 & \times & \cdots & \times & \times \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \times & \cdots & \times & \times
\end{array}\right]
$$

eliminate all entries below a_{11}.

then – – ———— a_{22}

then – ——— a_{33}

;

$$
\left[\begin{array}{ccccc|c}
\times & \times & \times & \times & \times & \times \\
 & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \vdots \\
 & 0 & & \times & \times & \vdots \\
 & & & & \times & \times
\end{array}\right]
$$

towards diagonal matrix

$$
\left[\begin{array}{ccccc|c}
\times & & & & & \times \\
 & \times & & 0 & & \times \\
 & & \times & & & \vdots \\
 & 0 & & \times & & \vdots \\
 & & & & \times & \times
\end{array}\right]
$$

$$
\begin{cases} 2x_1 = 6 \\ 3x_2 = 7 \end{cases}
$$

scale to get I.

$$
\left[\begin{array}{ccccc|c}
1 & & & & & \times \\
 & 1 & & 0 & & \times \\
 & & \ddots & & & \vdots \\
 & 0 & & 1 & & \vdots \\
 & & & & 1 & \times
\end{array}\right]
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \vdots \\ \times \end{bmatrix}
$$

(Think: does it always work?)

Theory:

Does this process always work?

Next. The current version of GE works under a condition.

Guarantee.

Claim For $n \times n$ system, if in the GE process,

Condition diagonal entries are all nonzero,

then we get the unique solution

in the end.

Part II Elementary Matrix

Strang's book Sec 2.3

—Three elementary matrices
—How to derive them → next time
—Permutation matrix → next time

Question

Goal: Better understand GE for solving a **square linear system**

Question:

Given a square linear system, after implementing the first step in GE (**forward elimination**), what happen to the **coefficient matrix**?

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \vdots & \ddots & \cdots & \vdots \\ \times & \times & \times & \times \end{bmatrix} \longrightarrow ?$$

Matrix Operation (row operation).

Start:

$$\begin{array}{c} \\ \end{array}\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right]\begin{array}{l} \to R_1 \\ \to R_2 \end{array}$$

$(R_2 \leftarrow) R_2 - R_1 \cdot 2$

Write augmented matrix for rest

next step.

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right]$$

$$\begin{array}{ll} R_2 & \leftarrow [2, 4 \mid 38] \\ -2R_1 & -2[1, 1 \mid 12] \end{array}$$

$$\begin{array}{ccc} 2-2\cdot 1 & 4-2\cdot 1 & 38-2\cdot 12 \\ \| & \| & \| \\ 0 & 2 & 14 \end{array}$$

Relate this to standard matrix operation.

Answers Yes, matrix multiplication.

Key claim

Row operation

$\Longleftrightarrow$ matrix multiplication

Review of Elementary Row Operations

Definition (Elementary Row Operations)

(1) [**Interchange**] Swap the positions of two rows

(2) [**Multiplication**] Multiply a row by a **non-zero** scalar

(3) [**Addition**] Add to one row a scalar multiple of another

The operations preserve solutions

Performing the operations above and creating a new system will not vary the solution set of the original system!

Can we use matrix multiplication to represent GE?

Row-Operation 3: via Row-Form of Linear System

Operation 3: Substract Eq. (1) * α - Eq. (2).
Can we represent it using matrix?

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \text{ where } A_1, A_2 \in \mathbb{R}^{1\times 2}.$$

$x \in \mathbb{R}^{2\times 1}$

$$Ax = b \Leftrightarrow \begin{cases} \underbrace{A_1}_{1\times 2} \underbrace{\vec{x}}_{2\times 1} = \underbrace{b_1}_{1\times 1} \\ A_2 \vec{x} = b_2 \end{cases}$$

$$\left[\begin{array}{c|c} A_1 & b_1 \\ A_2 & b_2 \end{array}\right]$$

Substract Eq. (1) * α - Eq. (2), get a new linear system

$$\begin{cases} A_1 \vec{x} = b_1 \\ (-\alpha A_1 + A_2)\,\vec{x} = b_2 - \alpha b_1 \end{cases} \qquad \left[\begin{array}{c|c} A_1 & b_1 \\ -\alpha A_1 + A_2 & -\alpha b_1 + b_2 \end{array}\right]$$

Matrix Representation of Operation 3

$$\left[\begin{array}{c|c} A_1 & b_1 \\ A_2 & b_2 \end{array}\right] \xrightarrow[R_2-\alpha R_1]{} \left[\begin{array}{c|c} A_1 & b_1 \\ -\alpha A_1 + A_2 & -\alpha b_1 + b_2 \end{array}\right]$$

Claim $\exists$ matrix $\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$, s.t.

$$\begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} \overbrace{A_1}^{1\times 2} \\ \overbrace{A_2}^{1\times 2} \end{bmatrix} = \begin{bmatrix} \overbrace{A_1}^{1\times 2} \\ -\alpha A_1 + A_2 \end{bmatrix}.$$

$A_1 + 0 \cdot A_2 = A_1$

$-\alpha \cdot A_1 + 1 \cdot A_2 = -\alpha A_1 + A_2$.

"Matrix operation $\xrightarrow[]{R_2 - \alpha R_1}$" applied to A $\Leftrightarrow$ multiply A by matrix $\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$

denote $E_{-\alpha R_1 + R_2}$.

Example: 3x3 matrix

$$A \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \xrightarrow{R_3 + 2R_1} \begin{bmatrix} A_1 \\ A_2 \\ 2A_1 + A_3 \end{bmatrix}$$

$$\| $$

$$\begin{array}{r} \\ \\ \text{3rd row} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

1st col.

Claim.

Applying row operation $R_3 + 2R_1$ to A

$$\Updownarrow$$

multiply A by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Operation 3: Add a Scaled Row

Add a Scaled Row to Another

$(R_j \to \beta R_i + R_j)$ $\qquad$ $E_{\beta R_i + R_j}$

(3) The elementary matrix corresponding to elementary row operation 3 $(R_j \to \beta R_i + R_j)$ is (elementary matrix type III)

Block form?

$$E_{\beta R_i+R_j} = \begin{array}{cl} \begin{bmatrix} \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & & & & \\ & 1 & & & \\ & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & & \\ & \beta & & 1 & \\ & & & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} \end{bmatrix} & \begin{array}{l} \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \end{array} \\ \quad i\text{th column} \qquad j\text{th column} & \end{array}$$

jth row

I_n

Operation 3: Add a Scaled Row

Row operation $\beta R_i + R_j \iff$ multiply A by $E_{\beta R_i + R_j}$

Add a Scaled Row to Another

Define ~~Suppose~~ $E_{\beta R_i + R_j} \in \mathbb{R}^{m \times m}$ (in previous page), the result of $E_{\beta R_i + R_j} A$ ($\alpha \neq 0$) is to multiply each element of ith row of matrix A by α (β), then add them into the jth row while keeping ith row unchanged.

Example

$$E_{2R_1+R_2}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ 2w+y & 2x+z \end{bmatrix}$$

$$E_{-2R_1+R_3}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ -2a_{11}+a_{31} & -2a_{12}+a_{32} \end{bmatrix}$$

Operation 1: Swap Rows

Swap Two Rows

$(R_i \leftrightarrow R_j)$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} A_2 \\ A_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} A_1 \\ A_3 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Examples of Matrix Multiplication

Swap Two Rows

$$(R_i \leftrightarrow R_j)$$

(1) The elementary matrix corresponding to elementary row operation 1 $(R_i \leftrightarrow R_j)$ is (elementary matrix type I)

Block form?

$$E_{R_iR_j} = \begin{array}{cl}
\begin{bmatrix}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & & & & 1 & & & \\
& & & & 1 & & & & & & \\
& & & & & \ddots & & & & & \\
& & & & & & 1 & & & & \\
& & & 1 & & & & 0 & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{bmatrix}
&
\begin{array}{l}
\\ \\ \\ i\text{th row} \\ \\ \\ \\ j\text{th row} \\ \\ \\ \\
\end{array}
\\
\begin{array}{cc} i\text{th column} & \qquad j\text{th column} \end{array} &
\end{array}$$

Operation 1: Swap Rows

Swap Two Rows $(R_i \leftrightarrow R_j)$

Suppose $E_{R_iR_j} \in \mathbb{R}^{m\times m}$, $A \in \mathbb{R}^{m\times n}$, the result of $E_{R_iR_j}A$ is just to exchange the ith row and jth row of matrix A, $E_{R_iR_j}$ is also called the **row exchange matrix**.

$$E_{R_iR_j}^T = E_{R_iR_j}$$

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} y & z \\ w & x \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

Operation 2: Multiply a Row by a Nonzero

Multiply a Row by a Nonzero Scalar

$(R_i \to \alpha R_i(\alpha \neq 0))$

(2) The elementary matrix corresponding to elementary row operation 2 $(R_i \to \alpha R_i(\alpha \neq 0))$ is (elementary matrix type II)

Block form?

$$E_{\alpha R_i} = \begin{array}{cl} \left[\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & \alpha & & & \\ \hline & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array}\right] & \begin{array}{l} \\ \\ \\ ith\ row \\ \\ \\ \\ \end{array} \\ i\text{th column} & \end{array}$$

$[\; A_1$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \xrightarrow{\alpha \cdot R_2} \begin{bmatrix} A_1 \\ \alpha A_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Operation 2: Multiply a Row by a Nonzero

Multiply a Row by a Nonzero Scalar

Suppose $E_{\alpha R_i} \in \mathbb{R}^{m\times m}(\alpha \neq 0)$, the result of $E_{\alpha R_i}A$ is just to multiply each element of ith row of matrix A by α.

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ 3y & 3z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 2a_{31} & 2a_{32} \end{bmatrix}$$

Elementary Matrix

Definition (Elementary Matrix) 初等矩阵.

The matrices corresponding to a single **elementary row operation** are called **elementary matrices**.

For a given matrix A, performing elementary row operation for A is equivalent to premultiplying A by the corresponding elementary matrix.

Row op $\Longleftrightarrow$ matrix multiplication.

Matrix Multiplication for Elementary Row Operations

Elementary Matrix

Exercise

Are the following matrices elementary?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Elementary Matrix

Exercise

Are the following matrices elementary?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Question: What does the last matrix do if we multiply it by any matrix (with matched dimensions)?

Concluding Section

Summary Today

Write your summary below.

One sentence summary:

Detailed summary: