Lecture 07

Solving Linear System II: Matrix Inverse and LU Decomposition

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Today's Lecture

Today … Inverse and LU: solution of Ax = b for "good" system (No row exchange)

After this lecture, you should be able to

1. Derive elementary matrix

2. Apply definition of inverse and properties of inverse

3. When no row exchange, write GE as product of elementary matrices

4. When no row exchange, derive LU decomposition and matrix inverse

Strang's book: Sec 2.4, 2.5

Part 0 Review

[5 mins]

Review: Allowable Operations on Rows

Definition (Elementary Row Operations) (初等行变换)

(1) [**Multiplication**] Multiply a row by a **non-zero** scalar

$$A_i \to$$

(2) [**Addition**] Add to one row a scalar multiple of another

$$A_i \to$$

(3) [**Interchange**] Swap the positions of two rows

$$\begin{pmatrix} A_i \\ A_j \end{pmatrix} \to$$

Example: 3×3 matrix

$$A \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \xrightarrow{R_3 + 2R_1} \begin{bmatrix} A_1 \\ A_2 \\ 2A_1 + A_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

3rd row

1st col.

<u>Claim.</u>

Applying row operation $R_3 + 2R_1$ to A

$\Updownarrow$

multiply A by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Recall: Add a Scaled Row

Claim: There exists a matrix $E_{\beta R_i + R_j}$ s.t.
applying $R_j \to \beta R_i + R_j$ (adding a Scaled Row to Another Row) to matrix A is equivalent to multiplying A by $R_j \to \beta R_i + R_j$

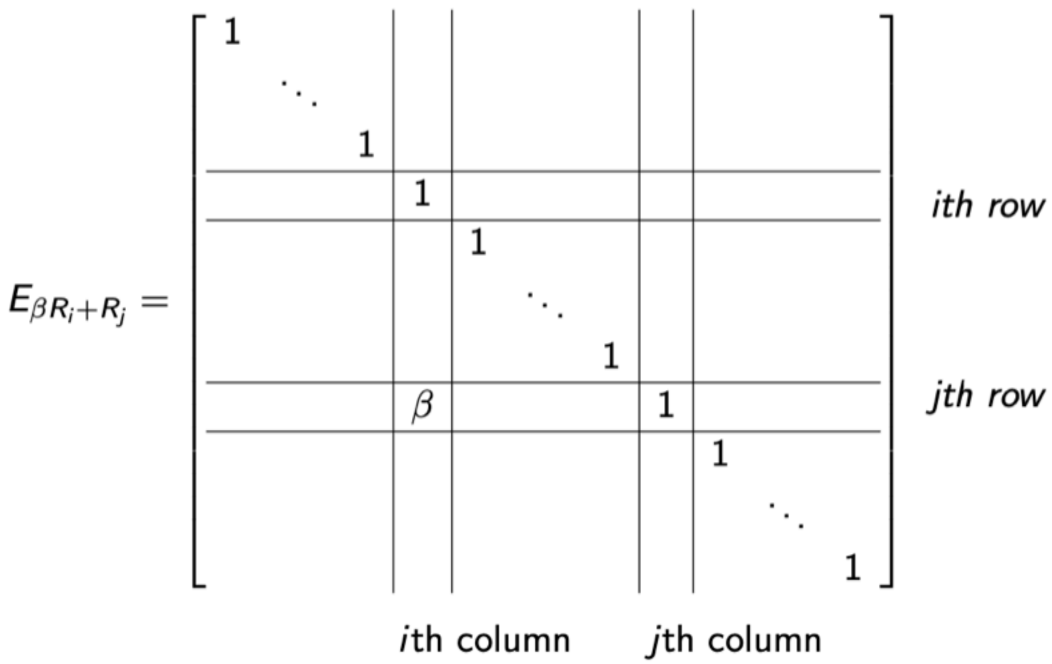
Proof: Consider the following matrix.
We can verify that $E_{\beta R_i + R_j}$ A is desired matrix after row operation. Done.

(3) The elementary matrix corresponding to elementary row operation 3 ($R_j \to \beta R_i + R_j$) is (elementary matrix type III)

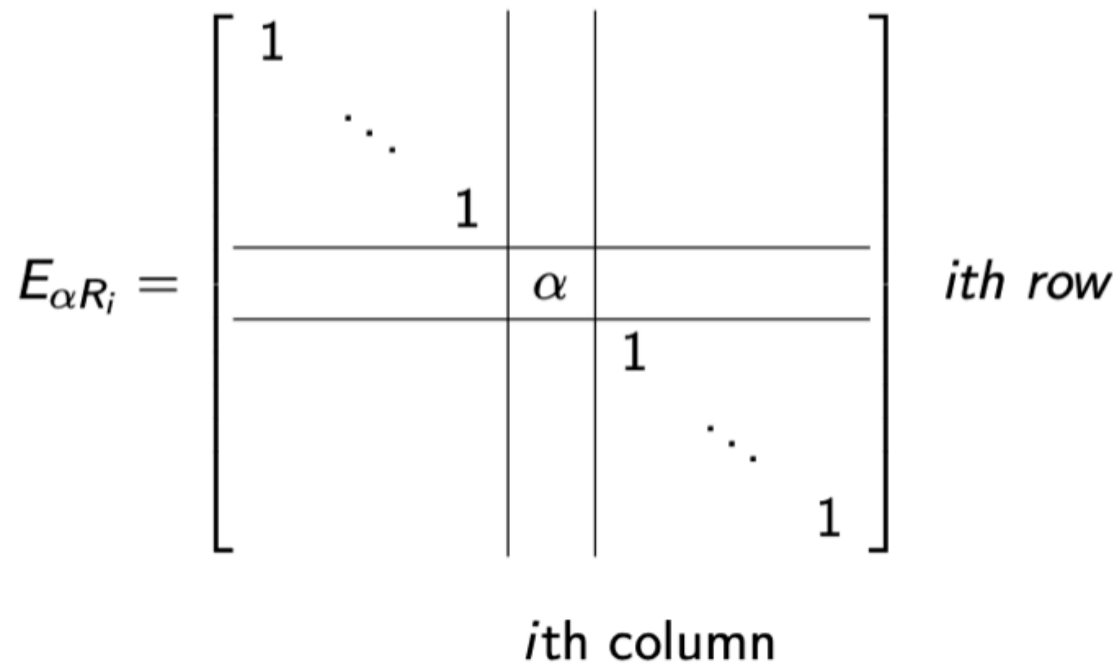
$$E_{\beta R_i + R_j} = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & \beta & & & & 1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ i\text{th row} \\ \\ \\ \\ j\text{th row} \\ \\ \\ \\ \end{matrix}$$

ith column $\quad$ jth column

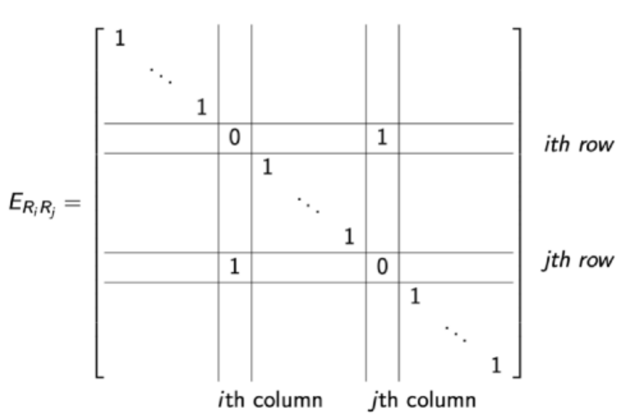
Recall: Three Elementary Matrices



Type-I



Type-II



Type-III

Part I Elementary Matrix (2)

Strang's book Sec 2.3

—How to derive elementary matrix

[10 mins]

How to Derive E_{R_i,R_j} from scratch?

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} e & f \\ c & d \\ a & b \end{bmatrix}.$$

What A can achieve this?

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} ?$$

Q: How to quickly write A s.t.

left-multiplying any matrix M

<==> swapping 1st & 3rd row?

How to Derive E_{R_i,R_j} from scratch?

Assume “swapping rows” can be achieved by matrix multiplication. Then what will it be like?

Trick: Apply it to identity matrix!

$$M \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Longrightarrow M =$$

Verify: Multiplying A by M indeed swaps rows of A.

Summarizing Logic of This Trick

What are we doing here?

Goal: Find A, s.t. $A \cdot M$ (where M is $n \times k$)

$\Longleftrightarrow$ applying $R_i \longleftrightarrow R_j$ to this matrix M. (*)

Q1 How to compute such a matrix A?

Answer: Apply $R_i \leftrightarrow R_j$ to I_n, to get a matrix $A \triangleq E_{R_iR_j}$.

This is just a definition. We need to verify it satisfies the desired property.

Q2. Does the matrix achieve the desired goal (*)?

Q2 i.e. need to verify:

$E_{R_iR_j} M$ = the matrix obtained by swapping R_i and R_j.

How to Derive Three Elementary Matrices?

Assume “elementary row operations” can be achieved by matrix multiplication.

Then what will they be like?

Trick: Apply it to identity matrix!

Elementary Matrix

Exercise

Are the following matrices elementary?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Elementary Matrix

Exercise

Are the following matrices elementary?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Question: What does the last matrix do if we multiply it by any matrix (with matched dimensions)?

Part II Matrix Inverse

Strang's book: Sec 2.5

—Definition
—Solution to $Ax = b$

[15 mins]

Motivation

What are the **benefits** of using matrix/vector notation?
—Simpler;
—Can utilize "rules" of simpler objects
e.g. AB = sum of outer products (inner product "rule")

Let's come back to linear systems.

$$ax = b \quad \text{<—>} \quad A\mathbf{x} = \mathbf{b}$$

What is the solution?

Question: **Can we write the solution in a similar way?**

Matrix Inverse

Definition (Inverse)

Suppose $A \in \mathbb{R}^{n \times n}$. If a matrix $B \in \mathbb{R}^{n \times n}$ satisfies

$$AB = BA = I_n,$$

then we say B is an inverse of A; denoted as $B = A^{-1}$.

Definition (Invertible)

Suppose $A \in \mathbb{R}^{n \times n}$. If there exists B such that $B = A^{-1}$, then we say A is invertible.

Questions: Does the inverse always exist?

Invertible Matrix

Question: Does the inverse always exist?

No! Consider n=1; you can find a counter-example.

Definition (Singular, nonsingular)

Suppose $A \in \mathbb{R}^{n \times n}$.
If A is invertible, then we say A is nonsingular.
If A is not invertible, then we say A is singular.

Example of Matrix Inverse

Example

(1) Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

then

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

Example of Matrix Inverse

Example (2) Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

Suppose

$$B = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = A^{-1},$$

then

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 2w+y & 2x+z \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case,

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \text{ is not invertible.}$$

Uniqueness of Matrix Inverse

Theorem (**Matrix inverse is unique**) Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Uniqueness of Matrix Inverse

Theorem (**Matrix inverse is unique**) Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Proof. Let B, C be the inverse of A, thus, $AB = BA = I, AC = CA = I$. Then $B = BI = BAC = IC = C$

Wording:

Previously: If $AB = BA = I_n$, then we say B is an inverse of A.

Now: If $AB = BA = I_n$, then we say B is the inverse of A.

Because of the uniqueness of inverse!

Invertible Matrix for Solutions of Linear Systems

Consider a linear system $A\mathbf{x} = \mathbf{b}$

What is x if A is invertible?

Theorem If A is invertible, then the linear system has a unique solution $x = A^{-1}b$.

Proof:

Specify requirements

Proof requires two things:

First, prove $A^{-1}b$ is a solution.

Second, prove $A^{-1}b$ is the unique solution; i.e.,
if y satisfies Ay = b, then y must be $A^{-1}b$

Invertible Matrix for Solutions of Linear Systems

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Proof:

Specify requirements

Verify requirements

Proof requires two things:

First, prove $A^{-1}b$ is a solution.

Second, prove $A^{-1}b$ is the unique solution; i.e., if y satisfies Ay = b, then y must be $A^{-1}b$

Verify two things:

Plug in: $A(A^{-1}b) = b$.

$$Ay = b \Rightarrow A^{-1}Ay = A^{-1}b$$
$$\Rightarrow y = A^{-1}b.$$

Part III Properties of Inverse

Strang's book Sec 2.5

Triangular matrices, products

[15 mins]

Matrix Inverse of Special Matrices

Property 9.1 (Inverse of Diagonal Matrix)

1) A diagonal matrix is invertible iff

2) The inverse of the diagonal matrix (if exists) is

Matrix Inverse of Special Matrices

Property 9.2 (Inverse of Triangular Matrix)

1) A triangular matrix is invertible iff

2) The inverse of an upper triangular matrix (if exists) is ______________________ matrix.

3) The inverse of a lower triangular matrix (if exists) is ___________________________ matrix.

Will be useful later this lecture!

Properties of Inverse: Inverse of Products

Theorem (**Matrix Inverse of Matrices Product**) Suppose A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Inverse Exercise

Exercise

$A_1, A_2, \ldots, A_k$ are real invertible matrices.

Question: What is $(A_1 A_2 \ldots A_k)^{-1}$?

Will be useful later this lecture!

Part IV GE Sequence and LU Decomposition: Overview

Strang's book Sec 2.6

[30 mins]

2 by 2 Example

Augmented matrix: $\begin{bmatrix} 1 & 1 & 12 \\ 2 & 4 & 38 \end{bmatrix}$

Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 12 \\ 2 & 4 & 38 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 12 \\ 0 & & \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \\ 0 & & \end{bmatrix}$$

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Just look at the coefficient matrix.

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

2 by 2 Example

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Gaussian elimination:

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Just look at the coefficient matrix.

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Express as matrix multiplication (just 1st step):

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \qquad E_{-3R_1+R_2} A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \end{bmatrix} \triangleq U.$$

2 by 2 Example

$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ GE step $\Leftrightarrow$ $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & \end{bmatrix} = U$.

Denote $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, then $E_{21}A = U$.

From $E_{21}A = U$, can we get an expression of A?

2 by 2 Example

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 1 \\ 0 & \end{bmatrix} = U.$$

Denote $E_{21} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, then $E_{21}A = U$.

From $E_{21}A = U$, can we get an expression of A?

Define $L =$ ___, then

$$A = LU.$$

Here, L is ____________matrix, and U is _______________ matrix.

3 by 3 Example

Coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

Gaussian elimination:

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

Express as matrix multiplication:

3 by 3 Example

Coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

Gaussian elimination:

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

Express as matrix multiplication:

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

3 by 3 Example: Expression of A

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \qquad E_3(E_2E_1A) = U\,.$$

Denote $M = E_3E_2E_1$, then $MA = U\,.$

From $MA = U$, can we get an expression of A?

Define $L =$ ___, then

$$A = LU\,.$$

Here, L is ____________matrix, and U is _______________ matrix.

3 by 3 Example: Verify L is Lower Triangular

Denote $M = E_3 E_2 E_1, \quad L = M^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$.

Claim 1: L is a lower triangular matrix.

3 by 3 Example: Verify L is Lower Triangular

Denote $M = E_3 E_2 E_1, \quad L = M^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$.

Claim 1: L is a lower triangular matrix.

Fact 1: E_3, E_2, E_1 are lower triangular matrices (Type-III elementary)

Fact 2: $E_3^{-1}, E_2^{-1}, E_1^{-1}$ are lower triangular matrices.

Mentioned in Slide "Shapes of Elementary Matrices?" In Part I.

3 by 3 Example: Verify L is Lower Triangular

Denote $M = E_3 E_2 E_1, \quad L = M^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$.

Claim 1: L is a lower triangular matrix.

Fact 1: E_3, E_2, E_1 are lower triangular matrices (Type-I elementary)

Fact 2: $E_3^{-1}, E_2^{-1}, E_1^{-1}$ are lower triangular matrices.

Property of inverse of triangular matrices.

Fact 3: Product of lower triangular matrices is lower triangular.

Homework 2 (though proved for upper triangular matrices).

n by n Good Case: LU Decomposition

Coefficient matrix A; square matrix.

Gaussian elimination (GE) (forward part; assume no row exchange**)**

$$A \to A_1 \to A_2 \ldots \to U$$

U is upper triangular.

Express GE as matrix multiplication:

$$E_k E_{k-1} \ldots E_1 A = U\,.$$

Express matrix A as matrix product:

$$A = (E_k E_{k-1} \ldots E_1)^{-1} U\,.$$

Denote $L = (E_k E_{k-1} \ldots E_1)^{-1}$**; it is lower triangular.**
since only involves Type-I (eliminating down) and Type-II elementary matrices
Then

$$A = LU\,.$$

Q&A

What are we doing here?

First, GE is just a bunch of matrix multiplications

$$A \to A_1 \to A_2 \ldots \to U$$

Second, multiply.

you can multiply matrices together

(Grouping them)

Third, shape.

it is a lower triangular matrix! (Wow!)

Claim of LU Decomposition: If No Row Exchange

Claim: Suppose A is a square matrix.
If there is no row exchange in the process of GE, then the matrix A can be decomposed as

$$A = LU,$$

where
—L is lower triangular;
—U is upper triangular.

Question: What if there are row exchanges?

Solve Linear Systems Million Times

—What if you need to solve $Ax = b_k$ for $k = 1,2,3,\ldots,10^{10}$?

Method 1:

Directly use GE to solve $Ax = b_k$.

Drawback: apply $E_k, E_{k-1}, \ldots, E_1$ for each system.
Computers memorize or apply all operations.

Method 2 (saving LU)

Compute. Compute L and U once.

Store. Store L and U.

Solve. Solve Ax = b by solving L y = b and U x= y

Premise: Solving Triangular System is Easy

Solving upper triangular system is easy:

Solving lower triangular system is also easy:

Concluding Section

Summary Today Write Your Own Summary

One sentence summary:

We learned

Detailed summary:

Summary Today Instructor’s summary

One sentence summary:

We learned

Detailed summary: