Lecture 08

Solving Linear System III: LU and PLU Decomposition

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Today's Lecture: Outline

Today … LU and PLU decomposition

1. LU Decomposition
2. Permutation matrix and PA = LU

After this lecture, you should be able to

1. Compute LU decomposition of simple matrices
2. Apply permutation in the GE process

Strang's book: Sec 2.5, 2.6

Part I Matrix Preparation:

Permutation Matrix and Inverse of Elementary Matrix

[15 mins]

Opening Practice

Q1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ is $E_{3R_1+R_3}$?

$3R_1$

Q2 What is A^{-1} ? $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

Q3. Does A^{-1} correspond to row operation?

$-3R_1 + R_3$ $\qquad E_{3R_1+R_3}^{-1} = E_{-3R_1+R_3}$

Q4 What do we learn from it?

Computing Inverse of $E_{3R_1+R_3}$

Naive method:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & f \end{bmatrix} \quad AB = BA = I \Rightarrow a, b, c, \ldots$$

Smarter method:

Shape of A: lower triangular

so inverse of A must be lower triangular.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

same row.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 3+b & c & 1 \end{bmatrix} \overset{\text{want}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a = 0, \; b = -3, \; c = 0$$

$$\Rightarrow B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Trick in Multiplication

$$\begin{bmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \alpha_3 \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} = \begin{bmatrix} \alpha_1 \vec{a} \\ \alpha_2 \vec{b} \\ \alpha_3 \vec{c} \end{bmatrix}$$

$$\begin{bmatrix} 20 & 5 & 60 \\ 0 & 1 & 0 \\ 100 & 75 & 37 \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} = \begin{bmatrix} \square \\ \vec{b} \\ \square \end{bmatrix}$$

Consider AB.

If in A, one row is the same as the row in identity matrix, then in the product AB, the corresponding row is the same as that row of B.

Q4: What to learn?

We have shown $E_{3R_1+R_3}^{-1} = E_{-3R_1+R_3}$.

What is the relation of $3R_1+R_3$ and $-3R_1+R_3$?

$$A = \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \xrightarrow{3R_1+R_3} \begin{bmatrix} \vec{a} \\ \vec{b} \\ 3\vec{a}+\vec{c} \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} = A$$

Apply two operators $\Longleftrightarrow$ doing **nothing**

They **are** reverse operators

If view matrix as operation

Inverse of matrix

$\Longleftrightarrow$ reverse operation

Reverse Operations in Daily Life

Reversable Operations

Turn off / on light

Wear / take off shoes.

Sleep / wake up.

Non-reversable operation

break a light bulb

burn shoes

What Is the Inverse of an Elementary Matrix?

Example

$A^{-1} = A$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $R_2 \leftrightarrow R_1$

$3R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ $\frac{1}{3}R_2$

$3R_1 + R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ $-3R_1 + R_3$

Question: What Is the Inverse of an Elementary Matrix?

Answer: Matrix corresponding to the reverse op.

Inverse of Elementary Matrix

tip: read title (important skill)

Theorem (Inverse of Elementary Matrices)

(1) $E_{R_iR_j}^{-1} = E_{R_iR_j}$, corresponding to the reverse row operation 1: $R_i \leftrightarrow R_j$.

(2) $E_{\alpha R_i}^{-1} = E_{\frac{1}{\alpha}R_i}$ $(\alpha \neq 0)$, corresponding to the reverse row operation 2: $R_i \to \frac{1}{\alpha}R_i$.

(3) $E_{\beta R_i + R_j}^{-1} = E_{-\beta R_i + R_j}$, corresponding to the reverse row operation 3: $R_j \to -\beta R_i + R_j$.

Remark. The inverse of the elementary matrices corresponding to the reverse row operations and belong to the same type of elementary matrices.

What are the Shapes? Triangular, diagonal or none?

Type I: Not triangular

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Type I inverse: Not triangular

αR_2

Type II: Diagonal

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Type II inverse: Diagonal

Type III: Lower triangular Or Upper triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Type III inverse: Lower triangular Or Upper triangular.

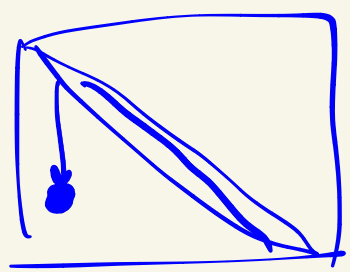
Same shape as the original matrix.

Add βR_1 to R_3 (use row-1 to eliminate row-3): lower triangular

If $i < j$, add βR_i to R_j (eliminate down, Step 1 of GE): lower triangular

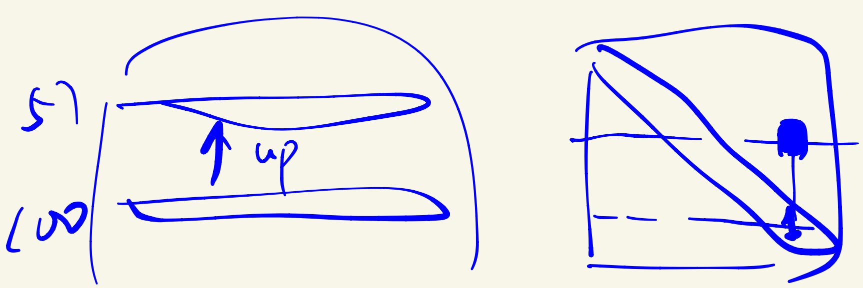
Otherwise: lower triangular.

$E_2 R_3 + R_7$: upp or low ?



GE: forward elimination ✓

$E_3 R_{100} + R_{57}$: upp ✓ or low ?



Exercise: Judgement

apply operation to I_n. (square)

Any elementary matrix is a square matrix.

True.

Any elementary matrix is invertible. True.

65% F 35% T.

Elementary Matrix

Exercise

Are the following matrices elementary?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Question: What does the last matrix do if we multiply it by any matrix (with matched dimensions)?

Permutation

A permutation π is a bijection from $\{1, \ldots, n\}$ to itself.

Expression as a vector: $\pi = (\pi(1), \ldots, \pi(n))^T$.

There are $n!$ permutations of n elements.

Matrix P_π is obtained by reordering the rows of I in the order $(\pi(1), \ldots, \pi(n))^T$.

- Let $\mathbf{e}_i$ be the ith row of I.
- The ith row of P_π is $\mathbf{e}_{\pi(i)}$.

$$\pi = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \quad P_\pi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

permutation of $\{1,2\}$: $(1\ 2)^T$, $(2\ 1)^T$

permutation of $\{1,2,3\}$: $(1,2,3)^T$, $(1,3,2)^T$, $(2,1,3)^T$
$(2,3,1)^T$, $(3,1,2)^T$, $(3,2,1)^T$

记号 Notation P_π

If $\pi = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $P_\pi = P_{\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

If $\pi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $P_\pi = P_{\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

If $\pi = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $P_\pi = P_{\left[\begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Permutation Vector via Swapping Entries

Claim Permutation vector can be obtained by swapping variables multiple times

$$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \xrightarrow[\text{many times}]{\text{swap entries}} \begin{bmatrix} \pi(1) \\ \vdots \\ \pi(n) \end{bmatrix} \text{ any permutation.}$$

e.g. How to obtain $\begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$ by swapping entries?

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \begin{matrix} 4 \\ 2 \\ 3 \\ 1 \end{matrix} \quad \begin{matrix} 4 \\ 1 \\ 3 \\ 2 \end{matrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

Permutation Matrix via Swapping Rows

<u>Claim</u> Permutation matrix can be obtained by swapping rows multiple times.

e.g. $\pi = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, $P_\pi = \begin{bmatrix} a_4 \\ a_1 \\ a_3 \\ a_2 \end{bmatrix}$, where a_i is the i'th row of $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.

$$I = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} a_4 \\ a_2 \\ a_3 \\ a_1 \end{bmatrix} \xleftrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} a_4 \\ a_1 \\ a_3 \\ a_2 \end{bmatrix}$$

Permutation Matrix via Elementary Matrices

Property (Permutation Matrix)

For a permutation matrix P, it can always be decomposed into a multiplication of finite number of row exchange matrices $E_{R_{i_k} R_{j_k}}$ (corresponding to the row exchange $R_{i_k} \leftrightarrow R_{j_k}$), i.e.

$$P = E_{R_{i_k} R_{j_k}} \cdots E_{R_{i_2} R_{j_2}} E_{R_{i_1} R_{j_1}}$$

①

permutation

product of $E_{R_i R_j}$'s

②

Example

each E: exchange rows once

exchange rows multiple times

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = E_{R_2 R_4} E_{R_1 R_3}$$

Permutation Matrix

Definition (Permutation Matrix)

A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

Remark A permutation matrix can be obtained by reordering the rows of the identity matrix.

Example

$$\pi = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$I \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_4 \\ a_1 \\ a_2 \end{bmatrix}$$

(3th, 4th, 1st, 2nd)

Thus $P = E_{R_1R_3} E_{R_2R_4}$

Part II Computing LU Decomposition

Strang's book Sec 2.6

[20 mins]

“Row Operation <==> Matrix Multiplication” in GE

We have learned two things:

(S1) (A single) elementary row operation
Is (a single) matrix multiplication. [Lec 06]

(S2) GE is a bunch of elementary row operations. [Lec 05]

(Corollary) by Syllogism 三段论
(S3) GE is a bunch of matrix multiplications.

(A is B) + (B is C) ⇒ (A is C)
(S1) (S2) (S3)

(S1) elementary row operation $\Longleftrightarrow$ matrix multiplication

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = B$$

$$\Longleftrightarrow \quad \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$E_{-2R_1+R_2} \, A = B.$$

(S2) GE is a bunch of elementary row operations.

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

upp   dig

Matrix Multiplications Lead to Lower Triangular

Two facts:

(S3) GE is just a bunch of matrix multiplications (last slide)

i.e. $E_k E_{k-1} \dots E_1 A = U$.

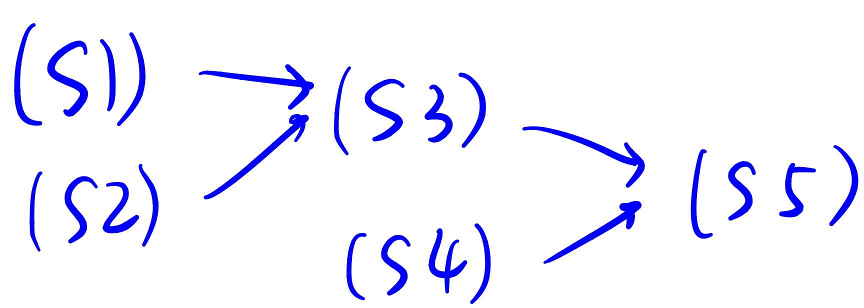
lower triangular matrix (if no row exchange)

(S4) Multiplication $E_k E_{k-1} \dots E_1$ is lower triangular.

(surprise) (Hw 2)

(Corollary)

(S5) A = (lower triangular) times (upper triangular) [Lec 07]



n by n Good Case: LU Decomposition

Coefficient matrix A; square matrix.

(S2) Gaussian elimination (GE) (forward part; assume no row exchange**)**

$$A \xrightarrow{Op\,1} A_1 \xrightarrow{Op\,2} A_2 \ldots \xrightarrow{Op\,k} U$$

U is upper triangular.

(S3) Express GE as matrix multiplication:

$$E_k E_{k-1} \ldots E_1 A = U.$$

Express matrix A as matrix product:

$$A = (E_k E_{k-1} \ldots E_1)^{-1} U.$$

(S4) Denote $L = (E_k E_{k-1} \ldots E_1)^{-1}$**; it is lower triangular.**

since only involves Type-I (eliminating down) and Type-II elementary matrices

Then

(S5) $$A = LU.$$

Computing LU: 2x2 Example

Example $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$. Write LU of it

Step 1 GE. $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \overset{\triangle}{=} U$

Step 2 matrix $\underbrace{E_{-2R_1+R_2}}_{\overset{\triangle}{=} E_1 \text{ (denote as, 记为)}} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{\overset{\triangle}{=} U \text{ (denote as, 记为)}}$. i.e. $E_1 A = U$.

Step 3 Compute L. $E_1^{-1} = E_{-2R_1+R_2}^{-1} = E_{2R_1+R_2} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Conclusion $A = L \cdot U$, when $L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Remark: You can verify $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Computing LU: 2x2 Exercise

Exercise $A = \begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix}$. LU of A.

Step 1. GE. $\begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix} \xrightarrow{2R_1+R_2} \begin{bmatrix} 2 & 3 \\ 0 & 14 \end{bmatrix} =: U$

Step 2: matrix $\underbrace{E_{2R_1+R_2}}_{=: E_1} A = U \Rightarrow A = E_1^{-1} U$

Step 3: Compute L: $L = E_{2R_1+R_2}^{-1} = E_{-2R_1+R_2} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Conclusion $A = LU$, where $L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 3 \\ 0 & 14 \end{bmatrix}$.

Simplification of Process

For experts, the above process is simplied to:

$$A = \begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix}$$

$$\downarrow 2R_1 + R_2$$

$$\begin{bmatrix} 2 & 3 \\ 0 & 14 \end{bmatrix} \triangleq U.$$

$$L = E_{2R_1+R_2}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$A = LU.$$

How to Compute LU Decomposition?

Step 1: Express GE as matrix multiplications

$$E_k E_{k-1} \dots E_1 A = U \,.$$

Step 2: Compute $L = E_1^{-1} E_2^{-1} \dots E_k^{-1}$.

Step 3: Write conclusion $A = LU$

Compute E_i's, U

Compute $\prod_i E_i^{-1} \triangleq L$

$\Downarrow$

$A = LU$

3 by 3 Example (Same slide as Lec 07)

Coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

Gaussian elimination:

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

Express as matrix multiplication:

1st entry below (1,1)

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

2nd below (1,1)

$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

1st below (2,2).

3 by 3 Example (from Lec 09)

Step 1 Gaussian elimination (forward elimination)

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1+R_3]{R_2 \to -\frac{1}{2}R_1+R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2+R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix} \triangleq U.$$

No row exchange; continue.

Step 2: Record elementary matrices.

$$E_1, E_2, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$E_{\times R_1+R_2}$, $E_{\times R_1+R_3}$ $E_{\times R_2+R_3}$.

3 by 3 Example (from Lec 09)

Step 1 Gaussian elimination (forward elimination)

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1+R_3]{R_2 \to -\frac{1}{2}R_1+R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2+R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix} \triangleq U.$$

No row exchange; continue.

Step 2: Record elementary matrices.

$$E_1, E_2, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Step 3: Compute L by (10.3).

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

positions fixed; values are ratios of entries

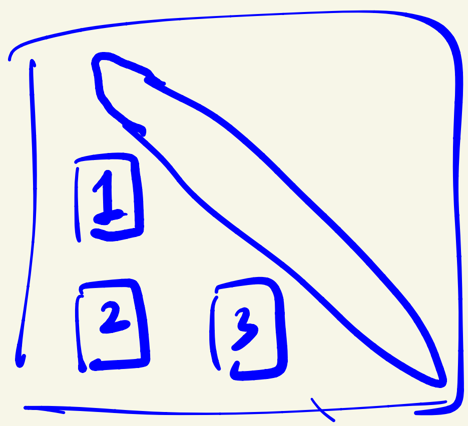
Step 4: Conclusion.

A has an LU decomposition A = LU, where L = blabla, U = blabla.

$$A = (E_3 \, E_2 \, E_1)^{-1} U$$

$E_{\alpha_3 R_2 + R_3}$ (E_3), $E_{\alpha_2 R_1 + R_3}$ (E_2), $E_{\alpha_1 R_1 + R_2}$ (E_1)

E_1, E_2, E_3 correspond to three entries to be eliminated.



Computing LU Decomposition

Algorithm (compute LU decomposition)

Step 1: GE

Step 2 Matrix $\tilde{E}_k \cdots \tilde{E}_1 A = U$.

Step 3: $L = (\tilde{E}_k \cdots \tilde{E}_1)^{-1}$

Step 4: $A = LU$.

Computing LU Decomposition

Algorithm (compute LU decomposition)

Step 1: **Forward elimination**.

Run forward elimination, till get upper triangular matrix U.

IF row exchange needed:

STOP and report: the algorithm fails to generate LU decomposition.

ELSE (no row exchange) Go to Step 2.

Step 2 Record elementary matrices.

Record elementary matrices $E_1, \ldots, E_k$ in Step 1.

Step 3: **Compute L.**

Compute $L = E_1^{-1} E_2^{-1} \ldots E_k^{-1} \quad (8.3)$

Remark:
Can use row operations for (8.3),

Step 4: Conclusion.

A has an LU decomposition A = LU,

where L = blabla (obtained in Step 3)

U = blabla (recorded in Step 1)

What if there is no LU decomposition?

You may wonder: what if there is row exchange?
(Assumption 2 does not hold)

In homework/exam:

—First, it probably will not happen if the problem asks you compute LU decomposition.

—Second, if it really happens in homework/exam, just say: there is no LU decomposition.

In practice:

—People can swap rows to perform PLU decomposition (next part)
—There may be other ways (beyond the class)

Part III PA = LU for General A

Strang's book Sec 2.6

Outline of This Part

Theorem (PLU decomposition)
Any square matrix A can be written as

$$P A = LU,$$

where L is lower triangular,
U is upper triangular;
P is a certain permutation matrix.

Outline:

—i) 2 by 2 matrix example;
—ii) 3 by 3 matrix example;
—iii) Summary: PA= LU.

2 by 2 Example

Augmented matrix: $\begin{bmatrix} 0 & 5 & 13 \\ 2 & 1 & -3 \end{bmatrix}$

$\begin{cases} 5x_2 = 13 \\ 2x_1 + x_2 = -3 \end{cases}$

Gaussian elimination:

$\begin{bmatrix} 0 & 5 & 13 \\ 2 & 1 & -3 \end{bmatrix}$

How to perform elimination step?

$E\ \underline{\alpha} R_1 + R_2$. what is α? $-\frac{1}{5}$.

$\begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 & (1) \\ a_{21} x_1 + a_{22} x_2 = b_2 & (2) \end{cases}$

To eliminate x_1 in (2), need $(1) \cdot \left(\frac{-a_{21}}{a_{11}}\right) + (2)$,

$a_{11} \neq 0$.

Now $a_{11} = 0$, what to do?

2 by 2 Example

Augmented matrix:

$$\begin{bmatrix} 0 & 5 & 13 \\ 2 & 1 & -3 \end{bmatrix}$$

$\begin{cases} 5x_2 = 13 \\ 2x_1 + x_2 = -3 \end{cases} \longrightarrow \begin{cases} 2x_1 + x_2 = -3 \\ 5x_2 = 13 \end{cases}$

Gaussian elimination:

$$\begin{bmatrix} 0 & 5 & 13 \\ 2 & 1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{2} & 1 & -3 \\ 0 & 5 & 13 \end{bmatrix}.$$

Swap the rows !!

The new $a_{11} \neq 0$.

2 by 2 Example

Augmented matrix: $\begin{bmatrix} 0 & 5 & 13 \\ 2 & 1 & -3 \end{bmatrix}$

Gaussian elimination:

$$\begin{bmatrix} 0 & 5 & 13 \\ 2 & 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & -3 \\ 0 & 5 & 13 \end{bmatrix}$$

Row exchange is needed!

Q: Unusual?

Just look at the coefficient matrix.

$$\begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$$

Answer: not triangular

Express as matrix multiplication (just 1st step):

$$A = \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix}$$

$$E_{R_1 \leftrightarrow R_2} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \triangleq U$$

2 by 2 Example

$$A = \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} \qquad E_{R_1 \leftrightarrow R_2} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \triangleq U.$$

Denote $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $P_1 A = U$.

Then $A = P_1^{-1} U$.

Here, P_1^{-1} is __not triangular__ matrix, and U is upper triangular matrix.

2 by 2 Example (cont'd)

$$A = \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} \qquad E_{R_1 \leftrightarrow R_2} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \triangleq U.$$

Denote $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $P_1 A = U$.

Then $A = P_1^{-1} U$.

Here, P_1^{-1} is ________________ matrix, and U is upper triangular matrix.

Is this an LU decomposition?

After re-ordering rows of A, the new matrix has LU decomposition;
i.e., ∃ permutation matrix P, s.t. $PA = LU$.
(In this example, $L = I, P = P_1^{-1}$.)

3 by 3 Example

Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ \times & \times & \times \end{bmatrix}$$

Gaussian elimination in matrix form (just coefficient matrix):

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

Row exchange is needed!

3 by 3 Example

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Row exchange is needed!

Define $P = E_{R_2 R_3}$, then $PE_2E_1A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$

Then $A = (PE_2E_1)^{-1}U = E_1^{-1}E_2^{-1}P^{-1}U$.

$A = L \cdot P^{-1} U$

3 by 3 Example

Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix}$$

Gaussian elimination in matrix form (just coefficient matrix):

$$E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

Row exchange is needed!

Define $P = E_{R_2R_3}$, then $PE_2E_1A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$

Then $A = (PE_2E_1)^{-1}U = E_1^{-1}E_2^{-1}P^{-1}U$.

But this is NOT the form of PA = LU. How to get it?

Idea: do all the row exchange at the beginning.

3 by 3 Example (cont'd)

Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix}$$

Define $P = E_{R_2 R_3}$, then $PA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix}$

Gaussian elimination in matrix form (just coefficient matrix):

$$E_2 E_1 (PA) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \triangleq U .$$

3 by 3 Example (cont'd)

Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix}$$

Define $P = E_{R_2 R_3}$, then $PA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix}$

Gaussian elimination in matrix form (just coefficient matrix):

$$E_2 E_1 (PA) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \triangleq U \, .$$

Then $PA = (E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} U \, .$

Define $L = E_1^{-1} E_2^{-1}$, then $PA = LU \, .$

n by n General Case: PA = LU

Coefficient matrix A: square matrix.

i) Gaussian elimination (GE) (allow row exchange; forward elimination)

$$A \xrightarrow{\text{Type-Ⅲ or none}} A_1 \xrightarrow{\text{Type-I}} A_2 \xrightarrow{\text{Type-Ⅲ or none}} A_3 \xrightarrow{\text{Type-I}} A_4 \ldots\ldots \to U$$

U is upper triangular.

if $a_{11}=0$,
row exchange
else:
do nothing

row op
to eliminate

n by n General Case: PA = LU

Coefficient matrix A; square matrix.

i) Gaussian elimination (GE) (allow row exchange; forward elimination)

$$A \xrightarrow{\text{Type-III or none}} A_1 \xrightarrow{\text{Type-I}} A_2 \xrightarrow{\text{Type-III or none}} A_3 \xrightarrow{\text{Type-I}} A_4 \ldots\ldots \to U$$

U is upper triangular.

ii) Express GE as matrix multiplication:

$$P_k E_k P_{k-1} E_{k-1} \ldots P_1 E_1 A = U,$$

Complicated

where $P_k, \ldots, P_1$ are row-exchange matrices or identity,
$E_k, \ldots, E_1$ are Type-II or Type-III elementary matrices.

iii) Extract row exchange matrices: define $P = P_k P_{k-1} \ldots P_1$.

$E_{\circ} P_{\circ} E_{\circ} P \cdots E_{\circ} P_{\circ} A = U.$

It's a somewhat complicated product.

Two things to proceed:

(i) (Magic) (to prove in hw 3).

you can swap $E_{\beta R_i + R_j}$ and P_{xx} (while changing index)

to get $\underbrace{(E_k \cdots E_1)}_{L^{-1}} \underbrace{(P_k P_{k-1} \cdots P_1)}_{P} A = U.$

$\Rightarrow L^{-1} P A = U \Rightarrow PA = LU.$

(ii) Product of row exchange matrix is a special type permutation matrix

n by n General Case: PA = LU

Coefficient matrix A; square matrix.

i) Gaussian elimination (GE) (allow row exchange; forward elimination)

$$A \xrightarrow{\text{Type-II or none}} A_1 \xrightarrow{\text{Type-I}} A_2 \xrightarrow{\text{Type-II or none}} A_3 \xrightarrow{\text{Type-I}} A_4 \ldots\ldots \to U$$

U is upper triangular.

ii) Express GE as matrix multiplication:

$$P_k E_k P_{k-1} E_{k-1} \ldots P_1 E_1 A = U,$$

where $P_k, \ldots, P_1$ are row-exchange matrices or identity,
$E_k, \ldots, E_1$ are Type-II or Type-III elementary matrices.

iii) Extract row exchange matrices: define $P = P_k P_{k-1} \ldots P_1$.

iv) Conduct GE on a new matrix PA.

"Magic": no row exchange is needed to obtain upper triangular matrix.

Claim (requires proof; skip): ∃ Type-II and Type-III matrices $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_t$ s.t.

$$\hat{E}_t \ldots \hat{E}_2 \hat{E}_1 (PA) = U .$$

Remark: $\hat{E}_i$'s are different from E_j's.

[This claim will be partially proved in HW 3]

n by n General Case: PA = LU

Coefficient matrix A; square matrix.

i) Gaussian elimination (GE) (allow row exchange; forward elimination)

$$A \xrightarrow{\text{Type-I or none}} A_1 \xrightarrow{\text{Type-II,III}} A_2 \xrightarrow{\text{Type-I or none}} A_3 \xrightarrow{\text{Type-II,III}} A_4 \ldots\ldots \to U$$

U is upper triangular.

ii) Express GE as matrix multiplication:

$$P_k E_k P_{k-1} E_{k-1} \ldots P_1 E_1 A = U,$$

where $P_k, \ldots, P_1$ are row-exchange matrices or identity,
$E_k, \ldots, E_1$ are Type-II or Type-III elementary matrices.

iii) Extract row exchange matrices: define $P = P_k P_{k-1} \ldots P_1$.

iv) Conduct GE on a new matrix PA**.**

"Magic": no row exchange is needed to obtain upper triangular matrix.

Claim (requires proof; skip)**:** ∃ Type-II and Type-III matrices $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_t$ s.t.

$$\hat{E}_t \ldots \hat{E}_2 \hat{E}_1 (PA) = U .$$

Remark: $\hat{E}_i$'s are different from E_j's.

v) Denote $L = (E_k E_{k-1} \ldots E_1)^{-1}$**, lower triangular. Then**

$$PA = LU .$$

PLU decomposition

$$PA = LU.$$

~~For any A,~~ after proper reordering of rows of A, it can be written as LU.

Concluding Section

Summary Today

Write your summary below.

One sentence summary:

Detailed summary:

Summary Today Instructor's summary

One sentence summary:

We learned how to compute LU decomposition, and PLU decomposition

Detailed summary:

1) Matrix preparation.
   - Inverse of permutation matrix: reverse operation
   - Permutation matrix (a) product of row exchange matrices; (b) reorder rows of I_n.
2) Compute LU decomposition
   - Works if no row exchange needed
   - Write E_i's, then compute $L = (E_k \cdots E_1)^{-1}$
3) PLU decomposition.
   - Any square matrix A satisfies $PA = LU$.