Lecture 9

Solving Square Linear System IV: Breakdown Cases and Solution Set

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Next Two Lectures: Key Questions

Consider a square linear system $A\mathbf{x} = \mathbf{b}$

Theorem If A is invertible, then the linear system has a unique solution $x = A^{-1}b$.

Question 1: When is A invertible?

Question 2: How to Express/Compute A^{-1}?

(if exists)

Will answer them in the coming lectures.

Q: If A is not invertible, what happens to linear system?

5-7 lectures

Today's Lecture: Outline

Today … Existence and expression/computation of inverse.

1. Breakdown Cases of Solving Square Linear System
2. Number of Solutions for Different Cases

Strang's book: Sec 2.2,

After this lecture, you should be able to

1. Tell a few breakdown cases for solving square linear system
2. Calculate number of solutions based on final form after elimination

Part 0 Preview

5 mins

Square System: What's New Compared to Primary School?

Primary school:

$$\begin{cases} x_1 + 2x_2 = 3, \\ 2x_1 - 3x_4 = 4. \end{cases} \qquad \begin{cases} 3x_1 + 2x_2 = 1, \\ 7x_1 - 5x_4 = 4. \end{cases} \qquad \begin{cases} x_1 - 2x_2 = 3, \\ 4x_1 + 7x_4 = -9 \end{cases} \qquad \ldots$$

Solve a 2*2 linear system? So easy!

Really?

Can you solve all 2×2 system?

(Imagine you're 10 yr old kid)

Square System: What's New Compared to Primary School?

Primary school:

$$\begin{cases} x_1 + 2x_2 = 3, \\ 2x_1 - 3x_4 = 4. \end{cases} \quad \begin{cases} 3x_1 + 2x_2 = 1, \\ 7x_1 - 5x_4 = 4. \end{cases} \quad \begin{cases} x_1 - 2x_2 = 3, \\ 4x_1 + 7x_4 = -9 \end{cases} \quad \ldots$$

Solve a 2*2 linear system? So easy!

All

Really?

process

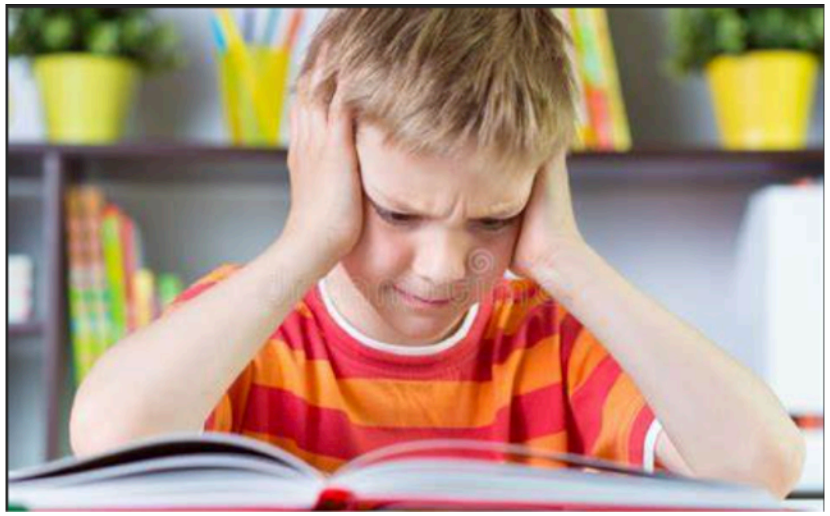
enumerate by reasoning

University:

$$\begin{cases} -x_1 + x_2 = 3, \\ 2x_1 - 2x_2 = 4. \end{cases}$$

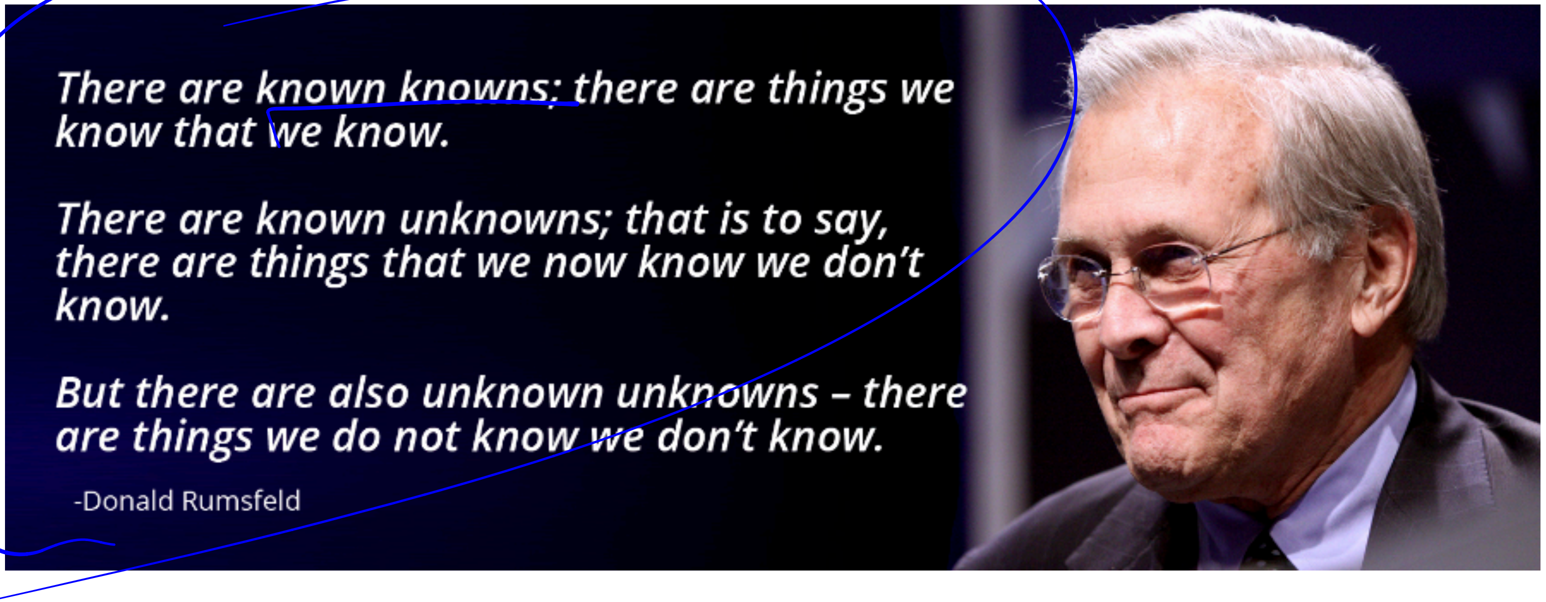
$$\begin{cases} -x_1 + x_2 = 3 \\ 0 \cdot x_1 + 0 \cdot x_2 = 10 \iff 0 = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 3 \\ 2 & -2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 1 & 3 \\ 0 & 0 & 10 \end{array}\right]$$



Conclusion The solution is NOT existent. (no solution

Unknown Unknowns



There are known knowns; there are things we know that we know.

There are known unknowns; that is to say, there are things that we now know we don't know.

But there are also unknown unknowns – there are things we do not know we don't know.

-Donald Rumsfeld

Knowns	**Known Knowns** Things we are aware of and understand.	**Known Unknowns** Things we are aware of but don't understand.
Unknowns	**Unknown Knowns** Things we understand but are not aware of.	**Unknown Unknowns** Things we are neither aware of nor understand.
	Knowns	Unknowns

Unknown unknowns

(for (most of the) 5th grade students):

Do I know how to solve all 2*2 linear systems?

They thought yes (they know). But actually no (they don't know).

find a solution.

Many 5th grade students don't even know what 'solve" really means:

It means "finding solution set" !

find all solution

find the set of solutions.

Part I “Breakdown” of Solving Square Systems

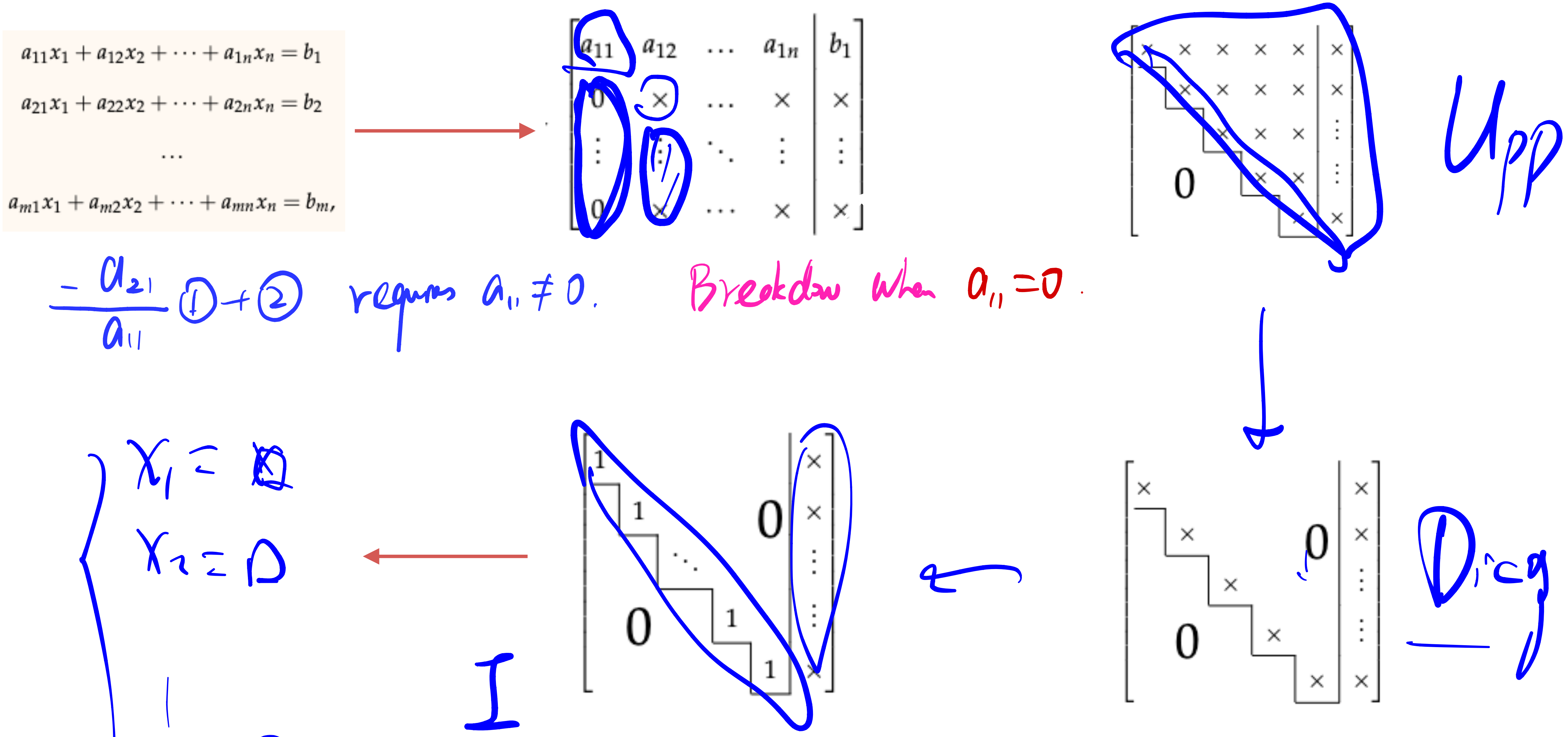
Sec 2.2, part “Breakdown of elimination”

20-30 mins

Review: Gaussian Elimination for "Good" Systems

Pipeline

eliminate entries below $a_{11}, a_{22}, \cdots$



$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & \times & \cdots & \times & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \times & \cdots & \times & \times \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & \times & \times & \times & \vdots \\ & 0 & & \times & \times & \vdots \\ & & & & \times & \times \end{array}\right]$$

Upp

$$-\frac{a_{21}}{a_{11}} ① + ②$$ requires $a_{11} \neq 0$.

Breakdown when $a_{11} = 0$.

$$\left[\begin{array}{ccccc|c} \times & & & & & \times \\ & \times & & 0 & & \times \\ & & \times & & & \vdots \\ & 0 & & \times & & \vdots \\ & & & & \times & \times \end{array}\right]$$

Diag

$$\left[\begin{array}{ccccc|c} 1 & & & & & \times \\ & 1 & & 0 & & \times \\ & & \ddots & & & \vdots \\ & 0 & & 1 & & \vdots \\ & & & & 1 & \times \end{array}\right]$$

I

$$\begin{cases} x_1 = \square \\ x_2 = \square \\ \quad | \\ x_n = \square \end{cases}$$

What if there are zero diagonal entries?

Summary of GE for Solving Square Systems

Step 1: Forward Elimination.
Perform elementary row operations and try to get an upper triangular matrix.

Step 2: **Backward substitution**

Perform elementary row operations and try to get a diagonal matrix.

Assumption 1 We can get a nonzero-diagonal triangular matrix after Step 1.

Claim 1 Under Assumption 1, we can get a diagonal matrix at the end of Step 2.

Corollary 1 Under Assumption 1, the system has a unique solution.

This assumption may not hold for some problems; will discuss later.

may breakdown

Pivot 枢纽.

Example

$$\begin{cases} x_1 + x_2 = 2, \\ 2x_1 - 2x_2 = 4. \end{cases}$$

Substract 2* Eq.1 from Eq. 2

$$\begin{cases} x_1 + x_2 = 2, \end{cases}$$

Goal.

Use ① to eliminate 2.

Matrix form:

$$\left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 2 & -2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & \times \\ 0 & -4 & \times \end{array}\right]$$

Pivot = ***first nonzero in the row that does the elimination***

The pivots are on the diagonal of the triangle after elimination.

Pivot: Examples

Lec 6.

(1,1)

$$A \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 1 & \underline{2} & 2 & 9 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

$$\Big\downarrow R_2 \leftarrow \underline{R_2} - R_1 ,\ R_3 \leftarrow \underline{R_3} - R_1$$

$$\hat{A} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$\Big\downarrow R_3 \leftarrow R_3 - \underline{R_2}$$

$$\hat{\hat{A}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

Step 1 Use $a_{11}=1$ to eliminate $a_{21}=1$, $a_{22}=1$,

so $\underline{a_{11}=1}$ is the 1st pivot

Forward

Step 2. Use $\hat{a}_{22}=1$ to eliminate $\hat{a}_{32}=1$,

so $\underline{\hat{a}_{22}=1}$ is the 2nd pivot.

Step 3 Use $\hat{\hat{a}}_{33}=1$ to eliminate $\hat{\hat{a}}_{23}=1$, $\hat{\hat{a}}_{13}=1$,

so $\underline{\hat{\hat{a}}_{33}=1}$ is the 3rd pivot.

Question: What if there are zeros?

Example 1 [Permanent failure with no solution.]

$$\begin{cases} x_1 + x_2 = 2, \\ 2x_1 + 2x_2 = 5. \end{cases}$$

Substract 2* Eq.1 from Eq. 2

$$\begin{cases} x_1 + x_2 = 2 \\ 0 = 1 \end{cases}$$

Matrix form:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

1st pivot is 1. ; 2nd pivot? no 2nd pivot,

The system has no second pivot.

$0 \notin$ pivot.

No solution

Zero is never allowed as a pivot!

Question: What if there are zeros?

Example 2 [Failure with infinitely many solutions .]

$$\begin{cases} x_1 + x_2 = 2, \\ 2x_1 + 2x_2 = 4. \end{cases}$$

$1+1=2$

$2\cdot 1+2\cdot 1=4$

Substract 2* Eq.1 from Eq. 2

$$\begin{cases} x_1 + x_2 = 2 \\ \quad 0 = 0 \end{cases}$$

1 pivot

Matrix form:

$$\left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \boxed{1} & 2 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

Any (x_1, x_2) satisfying $x_1+x_2=2$ is a solution:

$(1,1)$

$(2,0)$

$(1.5, 0.5)$

Conclusion Solution is any $(t, 2-t)$

∞-many solutions

English Lesson: Infinitely Many

noun

~~It has infinity solutions.~~ (grammarly not proper).

~~x_1, x_2 has infinity solutions.~~

Correct:

The system has "infinitely many" solutions.

adj

Informal: ∞ solutions or ∞-many solutions.

Question: What if there are zeros?

Example 3 [Temporary failure (zero). row exchge produces 2 pivots .]

$$\begin{cases} x_2 = 2, \\ 2x_1 + 2x_2 = 4. \end{cases}$$

Exchange two equations

Matrix form:

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} \boxed{2} & 2 & 4 \\ 0 & \boxed{1} & 2 \end{array}\right]$$

no ~~pivot~~

2 pivots

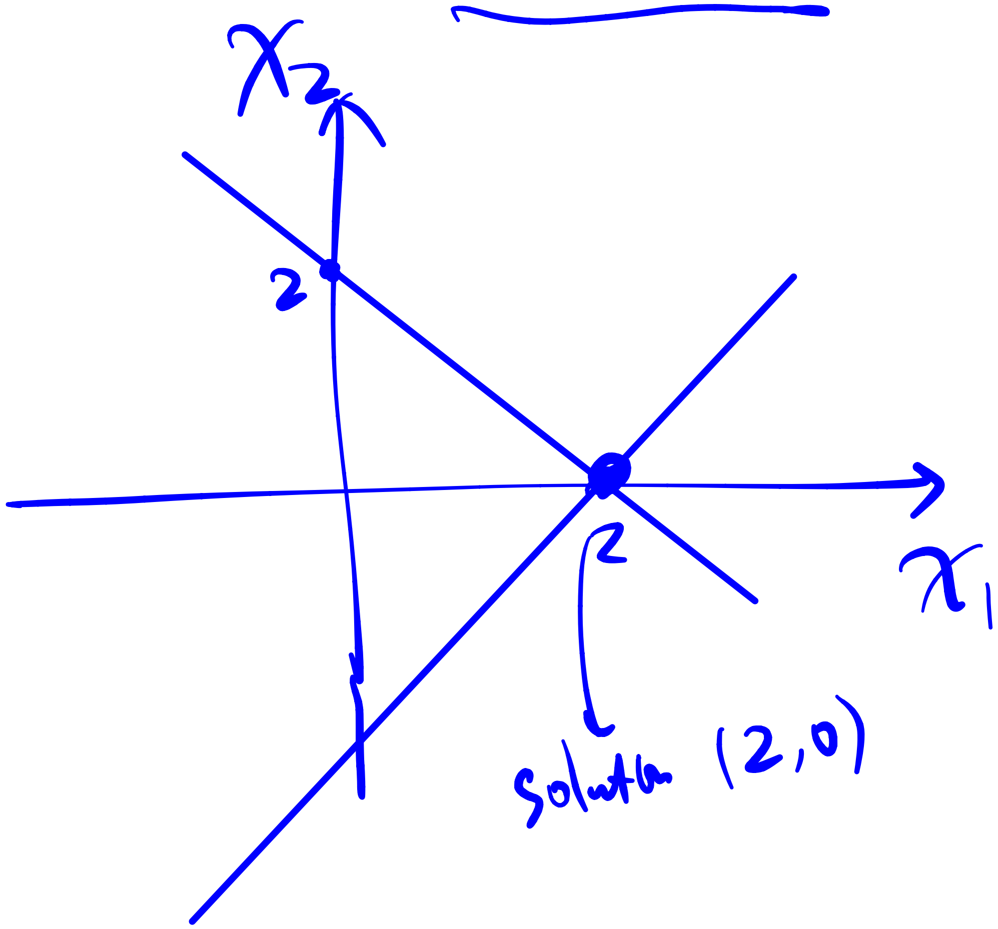
Visualization of Various Cases: 2 by 2 System

- Examples of solving 2×2 and 3×3 systems of linear equations
  Elimination!
- **Visualization of a 2×2 system of linear equations**

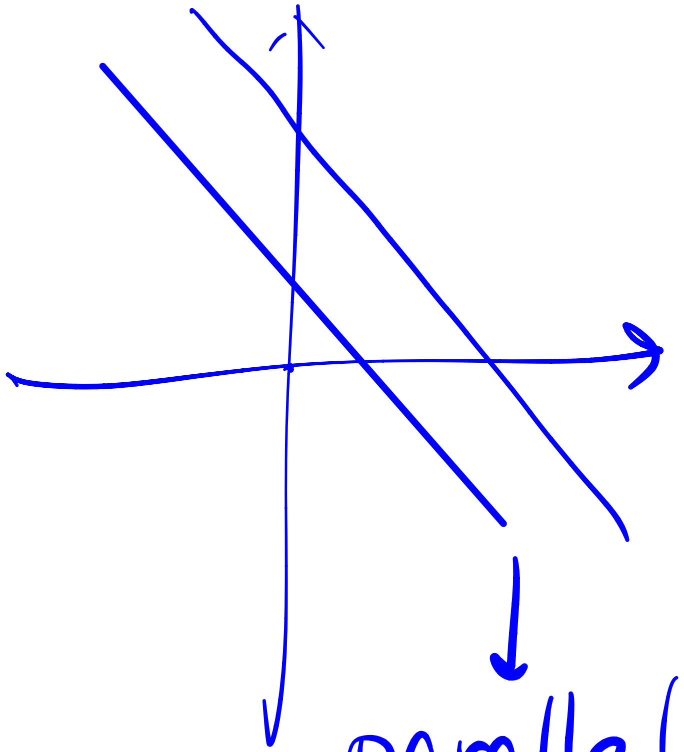
(i) $x_1 + x_2 = 2$
$x_1 - x_2 = 2$

(ii) $x_1 + x_2 = 2$
$x_1 + x_2 = 1$

(iii) $x_1 + x_2 = 2$
$-x_1 - x_2 = -2 \Leftrightarrow x_1 + x_2 = 2$

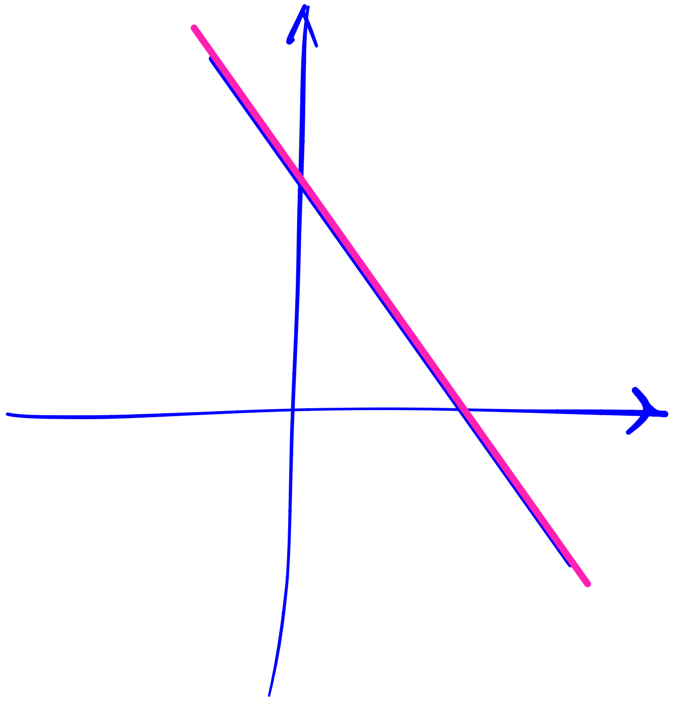


solution (2,0)



parallel

no solution

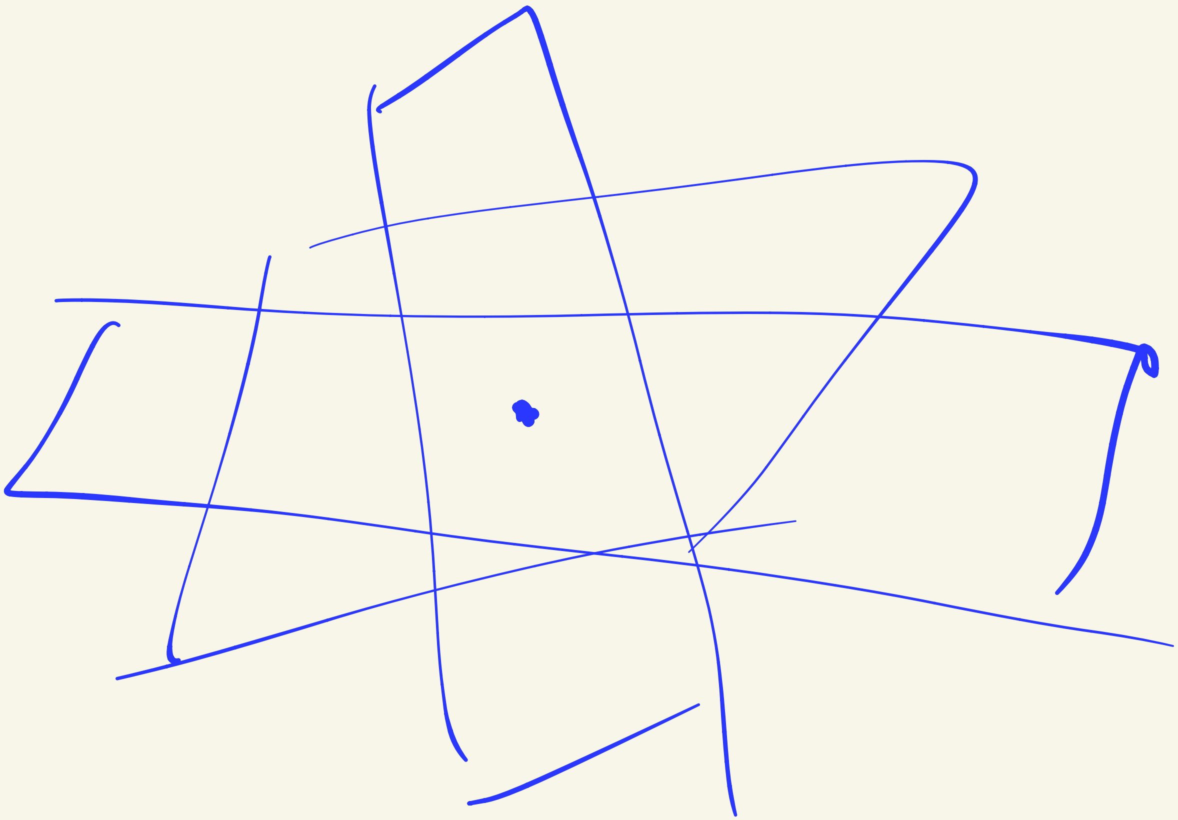


∞-many intersection

⇒ ∞-many solution

3D

4D



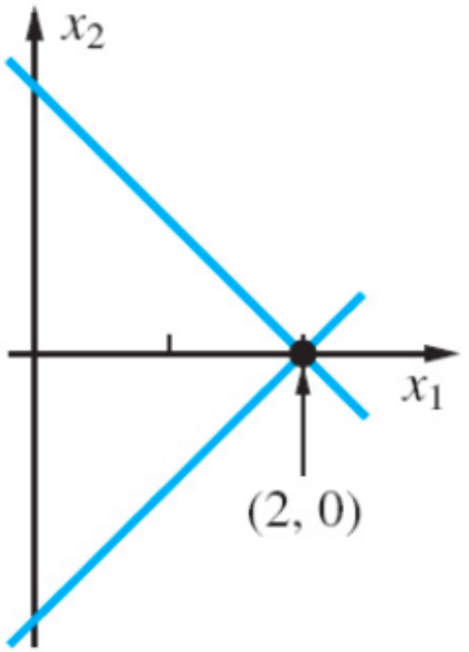
Visualization of Various Cases: 2 by 2 System

- Examples of solving 2×2 and 3×3 systems of linear equations Elimination!
- **Visualization of a 2×2 system of linear equations**

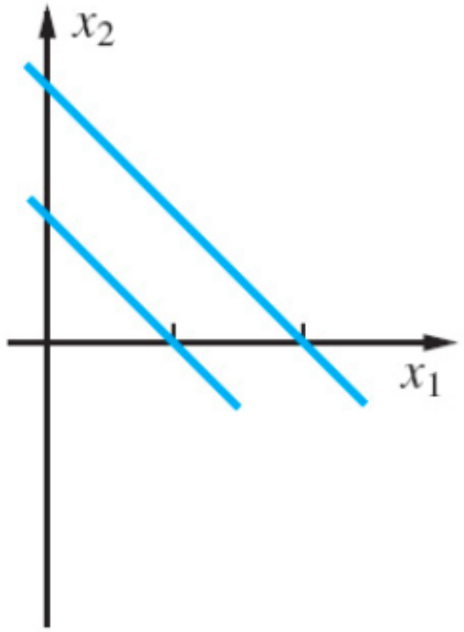
$$(i) \quad \begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned}$$

$$(ii) \quad \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}$$

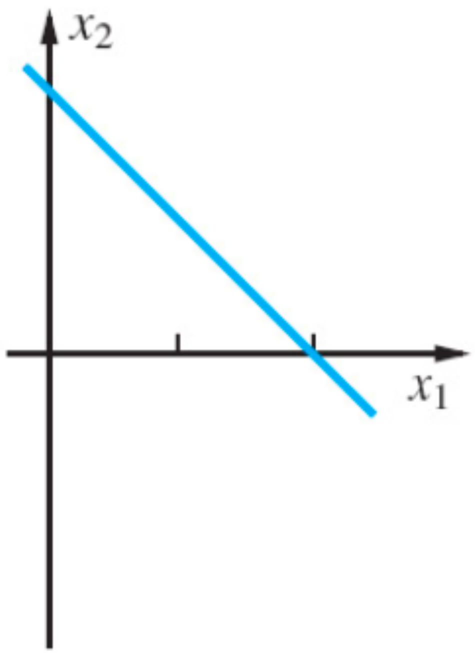
$$(iii) \quad \begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$



(i)



(ii)



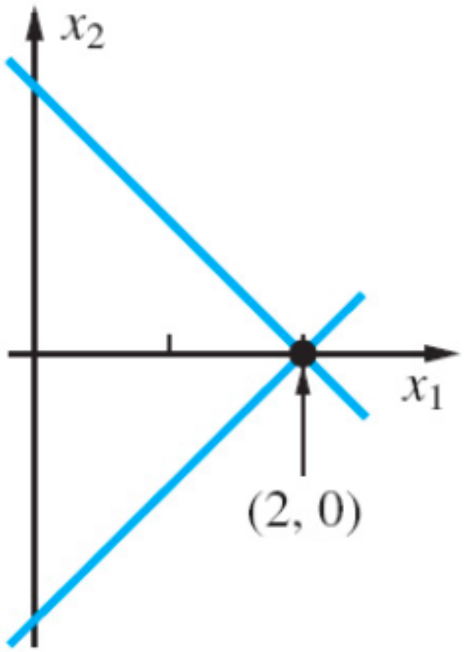
(iii)

Visualization of Various Cases: 2 by 2 System

- Examples of solving 2×2 and 3×3 systems of linear equations Elimination!
- **Visualization of a 2×2 system of linear equations**

(*i*) $x_1 + x_2 = 2$

$x_1 - x_2 = 2$

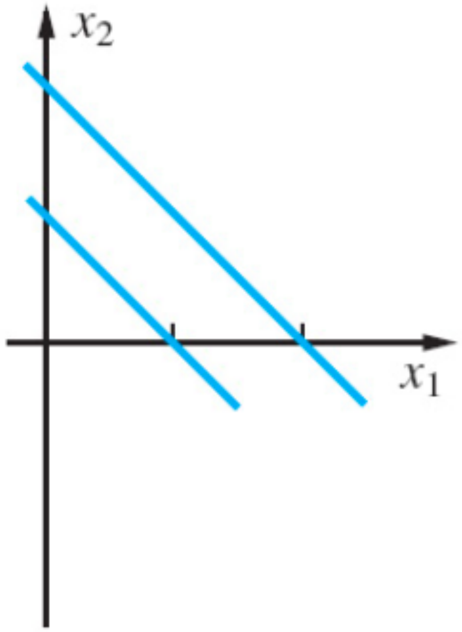


(i)

unique solution $x_1 = 2, x_2 = 0$

(*ii*) $x_1 + x_2 = 2$

$x_1 + x_2 = 1$

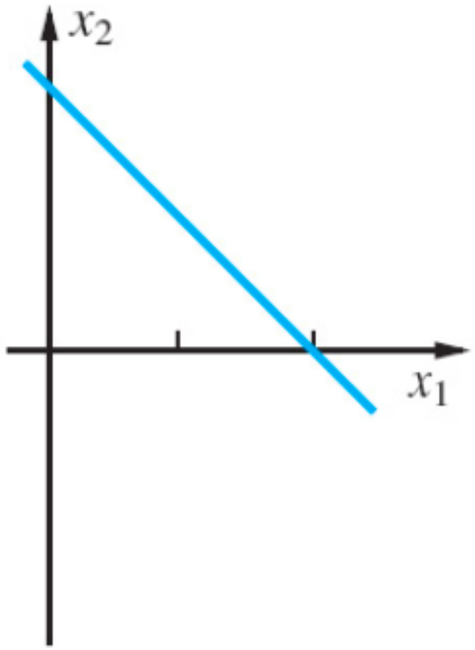


(ii)

no solution

(*iii*) $x_1 + x_2 = 2$

$-x_1 - x_2 = -2$



(iii)

infinitely many solutions

Judgement

A linear system can have exactly 3 solutions.

False.

must be 0, 1, ∞.

Solution Set

Definition (Solution and Solution Set)

A **solution** of an $m \times n$ linear system with variables $(x_1, \ldots, x_n)$ is a vector $(s_1, \ldots, s_n)$ such that if we let $x_i = s_i$ for all $i = 1, \ldots, n$ the m equations hold simultaneously.

The **solution set** of an $m \times n$ linear system is a set that contains all solution(s).

Examples of Solution Sets

Example (Solution Set)

$$\begin{aligned} 2x_1 + 3x_2 &= 3, \\ x_1 - x_2 &= 4 \end{aligned}$$

Examples of Solution Sets

Example (Solution Set)

$$2x_1 + 3x_2 = 3,$$
$$x_1 - x_2 = 4$$

The solution is (3,−1).

The solution set is $\{(3,-1)\}$ → double bracket

$$\vec{v} = (3,-1) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\{\vec{v}\}$$

Examples of Solution Sets

Example (Solution Set)

$$\begin{aligned} 2x_1 + 3x_2 &= 3, \\ x_1 - x_2 &= 4 \end{aligned}$$

The solution set is $\{(3, -1)\}$

$$\begin{aligned} 2x_1 + 3x_2 &= 3, \\ 4x_1 + 6x_2 &= 6 \end{aligned}$$

Examples of Solution Sets

Example (Solution Set)

$$\begin{aligned} 2x_1 + 3x_2 &= 3, \\ x_1 - x_2 &= 4 \end{aligned}$$

The solution set is $\{(3, -1)\}$

$$\begin{aligned} 2x_1 + 3x_2 &= 3, \\ 4x_1 + 6x_2 &= 6 \end{aligned}$$

The solution set is $\left\{ \underleftarrow{\left(t, \frac{3-2t}{3} \right)} \mid t \in \mathbb{R} \right\}$ (Infinitely many)

parameterized set

Will learn how to derive this set in later lectures

Part II Number of Solutions of Square System

Sec 2.2, part “Breakdown of elimination”

Have We Enumerated All Cases?

Q: We have seen a few breakdown cases.

Do we know how to handle ALL cases?

This section.

Try to enumerate all cases

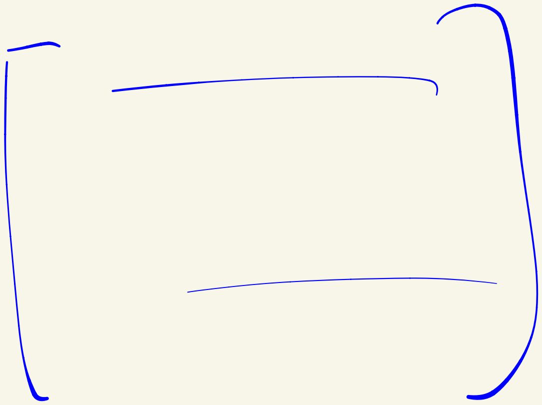
Check Your Knowledge Boundary: What's Unknown?

We have talked about 2×2 system

Discussed Breakdown cases. $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 4 \end{array}\right]$

What about bigger linear systems?

e.g. 10×10 system



Are you sure you can solve it?

Are you sure you can NOT solve it?

Two Cases for Square Systems

all possible cases of U

Coefficient matrix A; square matrix.

Gaussian elimination (GE) (both forward and backward)

$$A \to A_1 \to A_2 \ldots \to U \to \ldots B_1 \to \ldots\ldots \to D$$

D is diagonal?

Note: we don't always get a diagonal matrix.

End of forward elimination: Upper triangular

End of backward step

Case 1: Diagonal entries of U are nonzero.

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

✓ Diagonal matrix

_____ pivots

Case 2: Some diagonal entries of U are zero.

$$\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \\ & & & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-diagonal

"Non-diagonal" Matrix

structure:

$$\left[\begin{array}{c|c} I & \equiv \\ \hline 0 & 0 \end{array}\right]$$

_____ pivots

Two Cases During GE?

End of forward elimination: Upper triangular

End of backward step: [redacted]

final form of GE

Case 1: n pivots

$$\begin{bmatrix}1&*&*&*&*\\0&1&*&*&*\\\vdots&\vdots&\ddots&\ddots&*\\0&0&\cdots&1&*\\0&0&\cdots&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\\vdots&\vdots&\ddots&\ddots&0\\0&0&\cdots&1&0\\0&0&\cdots&0&1\end{bmatrix}$$

I

correspond to

$$\begin{cases}x_1=*,\\x_2=*,\\\vdots\\x_n=*,\end{cases}$$

Case 2: < n pivots.

I B

$$\begin{bmatrix}1&*&*&*&*&*&*\\0&1&*&*&*&*&*\\\vdots&\vdots&\ddots&\ddots&*&*&*\\0&0&\cdots&1&*&*&*\\0&0&\cdots&0&0&\cdots&0\\0&0&\cdots&0&0&\ddots&0\\0&0&\cdots&0&0&\cdots&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&\cdots&0&*&*&*\\0&1&\cdots&0&*&*&*\\\vdots&\vdots&\ddots&\ddots&*&*&*\\0&0&\cdots&1&*&*&*\\0&0&\cdots&0&0&\cdots&0\\0&0&\cdots&0&0&\ddots&0\\0&0&\cdots&0&0&\cdots&0\end{bmatrix}$$

0 0

correspond to

$$\begin{cases}x_1=*,\\\vdots\\x_k=*,\\0=c_{k+1},\\\vdots\\0=c_n.\end{cases}$$

or sth else ??

Remark: Rigorously speaking, there are other cases, which we discuss next.

Q: Can you always get

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} ?$$

Are there other cases ?

Answer: No other case, if allow column exchange.

New Case

Think: Can you make them $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$?

During Forward elimination:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{column exchange}} \begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Can we do it?

$$\left[\begin{array}{cccc|ccc} 1 & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * \\ \vdots & \vdots & \ddots & \ddots & * & * & * \\ 0 & 0 & \dots & 1 & * & * & * \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 3 & 0 \\ 0 & 0 & \dots & 0 & 0 & 4 & 5 \end{array}\right] \longrightarrow \begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ 0 - 0 \;\; 0 \;\; 3 \;\; 0 \\ 0 - 0 \;\; 0 \;\; 0 \;\; 0 \\ \text{———} \end{bmatrix}$$

$$\left[\begin{array}{c|ccc} I & & F & \\ \hline 0 & 3 & 0 & 0 \\ & 0 & 0 & 0 \\ & 4 & 0 & 5 \end{array}\right]$$

Another Example.

$$\left[\begin{array}{cc|ccc} 1 & 4 & \times & \times & \times \\ 0 & 1 & \times & \times & \times \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 5 & 8 \end{array}\right] \xrightarrow{R3 \leftrightarrow R4} \left[\begin{array}{cc|ccc} 1 & 4 & & B & \\ 0 & 1 & & & \\ \hline & & 0 & 4 & 7 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 5 & 8 \end{array}\right]$$

$$\Big\downarrow C3 \leftrightarrow C4$$

$$\left[\begin{array}{cc|ccc} 1 & 4 & & F & \\ 0 & 1 & & & \\ \hline & & 4 & 0 & 7 \\ & 0 & 0 & 0 & 0 \\ & & 5 & 0 & 8 \end{array}\right]$$

Effect of Swapping Columns: Concrete example

$$\begin{cases} x_1 + x_2 = 7 \\ 2x_1 + 4x_2 = 18 \end{cases} \longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\begin{cases} \hat{x}_2 + \hat{x}_1 = 7 \\ 2\hat{x}_2 + 4\hat{x}_1 = 18 \end{cases}$$

$$\downarrow$$

$$(x_1, x_2) = (5, 2)$$

exchange entries

$$\begin{cases} 1 \cdot y_1 + \cdot y_2 = 7 \\ 4 \cdot y_1 + 2 \cdot y_2 = 18 . \end{cases}$$

$$(y_1, y_2) = \cancel{(5, 2)}$$

$$= (2, 5)$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{swap columns}} \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

changing variable

(change subscripts 下标 of variables)

Swapping Columns: Symbolic Proof of 2x2 Case

Can we exchange columns? yes, if you do extra step

Consider $A = [\vec{a}_1, \vec{a}_2]$

$$Ax = 0 \Leftrightarrow \underline{x_1\vec{a}_1 + x_2\vec{a}_2} = 0$$

$\downarrow$

key property: commutative rule of addition

Swapping columns, get matrix

$$\hat{A} = [\,\vec{a}_2, \vec{a}_1\,]$$

Claim: Define $\hat{x} = [x_2, x_1]$ then

$$Ax = b \Leftrightarrow \hat{A}\hat{x} = b$$

Corollary: If $x = \boxed{(1,2)}$ is a solution of $Ax = b$, then

$\hat{x} = (2,1)$ is a solution to $\hat{A}\hat{x} = b$.

Reason. $\vec{a}_1 x_1 + \vec{a}_2 x_2 = \vec{a}_2 x_2 + \vec{a}_1 x_1$

i.e. $Ax = \hat{A}\hat{x}$

Eg 1

$$A \xrightarrow{C1 \leftrightarrow C2} \hat{A}, \quad \hat{A}y = b$$

$$x = (p_2, p_1) \xleftarrow{\text{swap } p_1, p_2} y = (p_1, p_2)$$

is the original solution

Eg 2

$$A \xleftrightarrow{C3 \leftrightarrow C5} \hat{A}, \quad \hat{A}y = b$$

$$x = (p_1, p_2, p_5, p_4, p_3) \xleftarrow[\text{swap } p3 \,\&\, p5]{} y = (p_1, p_2, p_3, p_4, p_5)$$

Claim Exchange column is OK, if you recover the original solution by swapping entries.

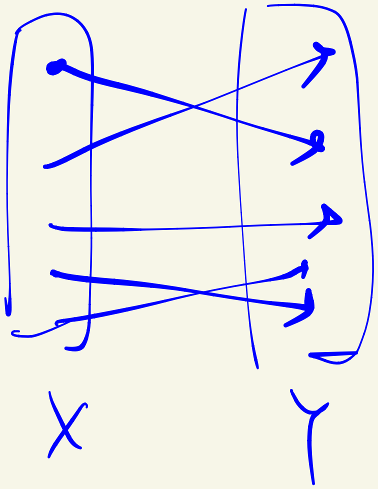
Swapping Columns: Result

Proposition: (swapping columns keep solution set structure)
Consider a linear system $Ax = b$.
Suppose $\hat{A}$ is obtained by exchanging columns of A multiple times.
Suppose the solution set of $Ax = b$ is X,
the solution set of $\hat{A}y = b$ is Y,
Then there exists a one-to-one mapping from X to Y.

mapping = function. $f: X \to Y$ means: for any $x \in X$, there's a $f(x) \in Y$.

1-1 mapping means: for any $y \in Y$, $\exists$ unique $x \in X$, s.t. $f(x) = y$

Corollary If $Ax = b$ has exactly p solution(s), where $p \in \{0, 1, \infty\}$,
Then $\hat{A}x = b$ also has exactly p solutions.

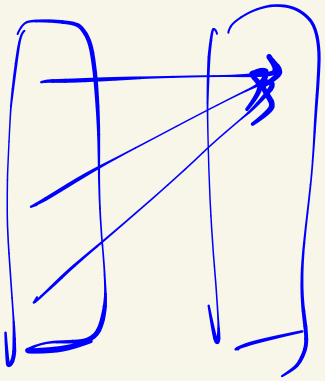


$f(x) = 2x$

is 1-1 mapping.

$\forall y$, can find unique x

s.t. $y = 2x = f(x)$



$f(x) = x^2$

Not 1-1 mapping

because: ① for $y < 0$

no x s.t. $f(x) = y$.

② for $y > 0$,

$\exists$ two numbers

$x = \sqrt{y}, -\sqrt{y}$ s.t. $x^2 = y$.

Claim If you all column exchange in GE, then for square matrix, you can obtain either

$$\left[\begin{array}{c|c} I & C \end{array}\right]$$

or

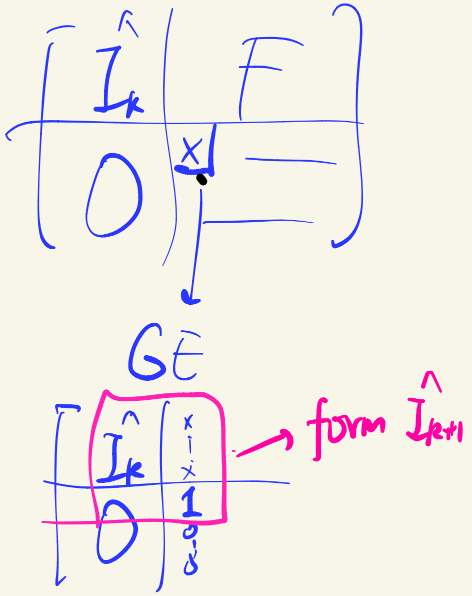
$$\left[\begin{array}{c|c} I_k & F \\ \hline 0 & \underbrace{0}_{(n-k)\text{ columns}} \end{array}\right] \Big\}\, (n-k)\text{ rows}$$

Proof Idea (Short version)

Let me call $\begin{bmatrix} 1 & \times & \times & \cdots & \times \\ 0 & 1 & \cdots & & \times \\ \vdots & & \ddots & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ as $\hat{I}_k$. (candidate for I_k).

Assume you have $\left[\begin{array}{c|c} \hat{I}_k & F \\ \hline 0 & B \end{array}\right]$ in the process. To prove: can obtain $\left[\begin{array}{c|c} \hat{I}_{k+1} & F_{k+1} \\ \hline 0 & B_{k+1} \end{array}\right]$.

$$\left[\begin{array}{c|c} \hat{I}_k & F \\ 0 & B \end{array}\right]$$



either $B = 0$ (done)

or $B \neq 0$

(use row & column exchange to make $(k+1, k+1)$ entry nonzero;

use row operation to make $(k+1)$-th column $\begin{bmatrix} \times \\ \vdots \\ \times \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. This is $\left[\begin{array}{c|c} \hat{I}_{k+1} & F_{k+1} \\ \hline 0 & B_{k+1} \end{array}\right]$

Proof Idea (Longer version)

Case 1. $B = 0$. Done.

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 4 & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \end{bmatrix}$$

top-left nonzero in B

Case 2. Some $B_{ij} \neq 0$. Pick the "first" nonzero entry

explain below

Informal. in the left-top corner.

Formal: scan the 1st colum of B, find 1st nonzero entry

$$B = \begin{bmatrix} 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 4 & \\ 0 & x & \end{bmatrix}$$

If success, do row exchange.

If fail. scan 2nd colum of B

- If success row & col exchange
- If fail. scan 3rd col.

⋮

NO other case missing! ☆

Namely: There is NO other case of the final form of GE with column exchange for square linear system.

Next: From the final form,

$$\left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array}\right] . \xrightarrow{\text{how to}} \text{write solution set.}$$

find form

Only Two Cases!!!!! (If Allow Column Exchange)

End of forward elimination: Upper triangular

End of backward step:

Case 1: n pivots

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

correspond to

$$\begin{cases} x_1 = \star\,, \\ x_2 = \star\,, \\ \vdots \\ x_n = \star\,, \end{cases}$$

I

$$\left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array}\right]$$

Case 2: < n pivots. (Allow column Exchange)

$$\begin{bmatrix} 1 & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * \\ \vdots & \vdots & \ddots & \ddots & * & * & * \\ 0 & 0 & \dots & 1 & * & * & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \dots & 0 & * & * & * \\ 0 & 1 & \dots & 0 & * & * & * \\ \vdots & \vdots & \ddots & \ddots & * & * & * \\ 0 & 0 & \dots & 1 & * & * & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

correspond to

$$\begin{cases} x_{i_1} = \dots\dots, \\ \vdots \\ x_{i_k} = \dots\dots, \\ 0 = c_{k+1}, \\ \vdots \\ 0 = c_n\,. \end{cases}$$

Remark 1: GE in the textbook does NOT perform column exchange.

Remark 2: If allowing swapping columns, then in the end, need to swap entries of obtained solution $(\hat{x}_1, \dots, \hat{x}_n)$ to obtain original solution.

For now, our goal is to study # of solutions, so we do not discuss details.

How to Continue to Write Solutions?

For Case 2, GE ends at the above two forms.

How to write solutions.

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Case 2: < n pivots. (Allow column exchange)

$$\begin{bmatrix} 1 & 0 & \dots & 0 & * & \dots & * \\ 0 & 1 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \ddots & * & \dots & * \\ 0 & 0 & \dots & 1 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \overset{\text{RHS}}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ c_{k+1} \\ \vdots \\ c_n \end{bmatrix}} \quad \text{correspond to} \quad \begin{cases} \hat{x}_1 = ?, \\ \vdots \\ \hat{x}_k = ?, \\ 0 = c_{k+1}, \\ \vdots \\ 0 = c_n . \end{cases}$$

Case 2a: If some $c_i \neq 0$, then no solution e.g. $0=3$, $0=7.5$, …

Case 2b: All $c_i = 0$.

$$\begin{cases} 0=0 \\ \vdots \\ 0=0 \end{cases}$$

Case 2b: Simple Example of 3 by 3

Case 2b: All $c_i = 0$..

Simple case:

$$\begin{array}{ccc|c} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \\ 1 & 0 & \alpha_1 & c_1 \\ 0 & 1 & \alpha_2 & c_2 \\ \hline 0 & 0 & 0 & 0 \end{array} \longrightarrow \begin{cases} \hat{x}_1 + \alpha_1 \hat{x}_3 = c_1 \\ \hat{x}_2 + \alpha_2 \hat{x}_3 = c_2 \\ 0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{x}_1 = c_1 - \alpha_1 \hat{x}_3 \\ \hat{x}_2 = c_2 - \alpha_2 \hat{x}_3 \end{cases}$$

It means: For any $\hat{x}_3 = t$ (free variable).

you can get $\begin{cases} \hat{x}_1 = c_1 - \alpha_1 t \\ \hat{x}_2 = c_2 - \alpha_2 t \end{cases}$ is solution.

For any t, $(c_1 - \alpha_1 t, c_2 - \alpha_2 t, t)$ is a solution

∞-many solutions

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ x_3 &= c_3 \end{aligned} \quad \rightarrow \text{unique}$$

free

$$\begin{aligned} x_1 &= c_1 + \boxed{\equiv}\, x_4 \\ x_2 &= c_2 + \boxed{\equiv}\, x_4 \\ x_3 &= c_3 + \boxed{\equiv}\, x_4 \end{aligned}$$

$x_4 \in \mathbb{R}$; ∞-many x_4 $\Rightarrow$ ∞-many solution

Remark

If original linear system is

$$\begin{cases} x_1 = 2 + x_4 \\ x_2 = 2 - x_4 \\ x_4 = 2 - 2x_4 \end{cases}$$

Then it has ∞-many solutions.

Case 2b: General Case Analysis

Case 2b: All $c_i = 0$..

Claim: Suppose $c_{k+1} = \ldots = c_n = 0$..
If $u_{ii} = 0$ for some i, then Ax =0 has __________ solution.

$$\left[\begin{array}{cccc|ccc|c}
1 & 0 & \ldots & 0 & * & \ldots & * & c_1 \\
0 & 1 & \ldots & 0 & * & \ldots & * & c_2 \\
\vdots & \vdots & \ddots & \ddots & * & \ldots & * & \vdots \\
0 & 0 & \ldots & 1 & * & \ldots & * & c_k \\
\hline
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & c_{k+1} \\
0 & 0 & \ldots & 0 & 0 & \ddots & 0 & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & c_n
\end{array}\right]$$

correspond to

$$\begin{cases}
\hat{x}_1 = ?, \\
\vdots \\
\hat{x}_k = ?, \\
0 = c_{k+1}, \\
\vdots \\
0 = c_n .
\end{cases}$$

$$\begin{cases}
\hat{x}_1 = c_1 + \boxed{?}\,\hat{x}_{k+1} + \cdots + \boxed{?}\,\hat{x}_n \\
\vdots \\
\hat{x}_k = c_k + \boxed{?}\,\hat{x}_{k+1} + \cdots + \boxed{?}\,\hat{x}_n
\end{cases}$$

∞-many solutions

From Matrix to Linear System

$$\begin{array}{cccccc|c}
\hat{x}_1 & \cdots & \hat{x}_k & \hat{x}_{k+1} & \cdots & \hat{x}_n & \\
1 & & 0 & \times & \times & \times & c_1 \\
 & \ddots & & \vdots & & & \vdots \\
 & & 1 & \times & \cdots & \times & c_k \\
\hline
0 & - & 0 & 0 & 0 - & 0 & c_{k+1} \\
 & & & & & & \vdots \\
0 & - & 0 & 0 & - & 0 & c_n
\end{array}$$

If $c_{k+1} = \cdots = c_n = 0$ $\longrightarrow$

$$\left\{\begin{array}{l}
\boxed{\hat{x}_1} + 0\cdot\hat{x}_2 + \cdots\ 0\cdot\hat{x}_k + \underline{\text{目}\,\hat{x}_{k+1} + \cdots + \text{目}\,\hat{x}_n} = c_1 \\
\hat{x}_2 + \quad \text{目}\,\hat{x}_{k+1} - \quad - \quad \hat{x}_n = c \\
\hat{x}_k + \text{目}\,\hat{x}_{k+1} \quad + \text{目}\,\hat{x}_n = c \\
\hline
0 = c_{k+1} \neq 0 \\
\vdots \\
0 = c_n
\end{array}\right.$$

From Linear System to Solution

$$\begin{cases} \hat{x}_1 + \boxed{\equiv}\,\hat{x}_{k+1} + \cdots + \boxed{\equiv}\,\hat{x}_n = c_1 \\ \quad\ddots \\ \quad\quad \hat{x}_k + \boxed{\equiv}\,\hat{x}_{k+1} + \cdots + \boxed{\equiv}\,\hat{x}_n = c_n \end{cases}$$

$$\Rightarrow \left| \begin{array}{l} \hat{x}_1 = c_1 - \boxed{\equiv}\,\hat{x}_{k+1} - \cdots - \boxed{\equiv}\,\hat{x}_n \\ \vdots \\ \hat{x}_k = c_n - \boxed{\equiv}\,\hat{x}_{k+1} - \cdots - \boxed{\equiv}\,\hat{x}_n. \end{array} \right.$$

$\hat{x}_{k+1}, \dots, \hat{x}_n$ are free variables.

For any choice of $\hat{x}_{k+1}, \dots, \hat{x}_n$, pick $\hat{x}_1, \dots, \hat{x}_k$ accordingly,

this gives a solution.

∞ choices of $x_{k+1}, \dots, x_n$, so ∞ solutions.

Summary of Success/Failure for Square Systems

Case 1: n pivots.

Unique solution.

We may have to exchange the equations (rows).

Case 2: For n by n systems, we do not get n pivots.

(allow col exchange) $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

Elimination leads to equations (besides $x_i = \star$)

0 = ☐ (nonzero number) [no solution(s)]

all 0 = 0 [∞ many solution(s)]

What you (should) Know by now

1) How to calculate
# of solutions for ANY square system,

---

2) How to write solution set of ANY square system,
(Suppose you know: for each "column exchange of matrix",
you need to perform "entry exchange of final solution")

[For "writing solution set", will have more lectures)

Gaussian Elimination for “Good” Systems

Problem: Solve n × n linear system of equations.

Preparation: Write the linear system as augmented matrix form.

Step 1:

Perform elementary row operations to get an upper triangular matrix.

Step 2:

Step 2.1 Perform elementary row operations to get a diagonal matrix.

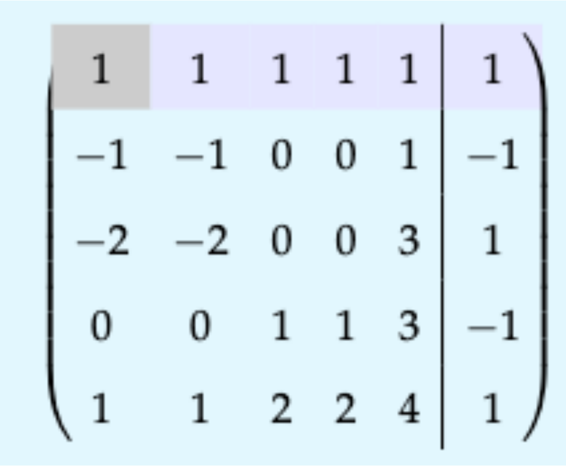
Step 2.2 If all diagonal entries are nonzero, (i.e. n pivots)

then perform “row multiplication” to get an identity matrix.

Finishing step: Write down the solution

(unique solution for this case).

Exercise



$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right)$$

Summary Today (write Your Own)

One sentence summary:

Detailed summary:

Summary Today (of Instructor)

One sentence summary:

We study

Detailed summary: